EXERCISE 2.1

Write the correct answer in each of the following:

1. Which one of the following is a polynomial?

(*a*) $\dfrac{x^2}{2}-\dfrac{2}{x^2}$ (*b*) $\sqrt{2x}-1$

(*c*) $x^2+\dfrac{3x^{\frac{3}{2}}}{\sqrt{x}}$ (*d*) $\dfrac{x-1}{x+1}$

Sol. (*a*) $\dfrac{x^2}{2}-\dfrac{2}{x^2}=\dfrac{x^2}{2}-2x^{-2}$

Second term is $-2x^{-2}$. Exponent of x^{-2} is -2, which is not a whole number. So, this algebraic expression is not a polynomial.

(*b*) $\sqrt{2x}-1=\sqrt{2}x^{\frac{1}{2}}-1$

First term is $\sqrt{2}x^{\frac{1}{2}}$. Here, the exponent of the second term, *i.e.,* $x^{\frac{1}{2}}$ is $\dfrac{1}{2}$, which is not a whole number. So, this algebraic expression is not a polynomial.

(*c*) $x^2+\dfrac{3x^{\frac{3}{2}}}{\sqrt{x}}=x^2+3x$

In this expression, we have only whole numbers as the exponents of the variable in each term. Hence, the given algebraic expression is a polynomial.

2. $\sqrt{2}$ is a polynomial of degree

(*a*) 2 (*b*) 0 (*c*) 1 (*d*) $\dfrac{1}{2}$

Sol. $\sqrt{2}$ is a constant polynomial. The only term here is $\sqrt{2}$ which can be written as $\sqrt{2}\ x^0$. So, the exponent of x is zero. Therefore, the degree of the polynomial is 0.

Hence, (*b*) is the correct answer.

3. Degree of the polynomial of $4x^4+0x^3+0x^5+5x+7$ is

(*a*) 4 (*b*) 5 (*c*) 3 (*d*) 7

Sol. The highest power of the variable in a polynomial is called the degree of the polynomial. In this polynomial, the term with highest power of x is $4x^4$. Highest power of x is 4, so the degree of the given polynomial is 4.

4. Degree of the zero polynomial

(a) 0 (b) 1

(c) Any natural number (d) Not defined.

Sol. Degree of the zero polynomial (0) is not defined.
Hence, (d) is the correct answer.

5. If $p(x) = x^2 - 2\sqrt{2}x + 1$, then $p(2\sqrt{2})$ is equal to

(a) 0 (b) 1 (c) $4\sqrt{2}$ (d) $8\sqrt{2}+1$

Sol. We have $p(x) = x^2 - 2\sqrt{2}x + 1$

$$\therefore \quad p(2\sqrt{2}) = (2\sqrt{2})^2 - 2\sqrt{2}\,(2\sqrt{2}) + 1$$
$$= 8 - 8 + 1$$
$$= 1$$

Hence, (b) is the correct answer.

6. The value of the polynomial $5x - 4x^2 + 3$, when $x = -1$ is

(a) -6 (b) 6 (c) 2 (d) -2

Sol. Let $p(x) = 5x - 4x^2 + 3$
Therefore, $p(-1) = 5(-1) - 4(-1)^2 + 3 = -5 - 4 + 3 = -6$
Hence, (a) is the correct answer.

7. If $p(x) = x + 3$, then $p(x) + p(-x)$ is equal to

(a) 3 (b) $2x$ (c) 0 (d) 6

Sol. We have $p(x) = x + 3$, then
$$p(-x) = -x + 3$$
Therefore, $p(x) + p(-x) = x + 3 + (-x + 3) = x + 3 - x + 3 = 6$
Hence, (d) is the correct answer.

8. Zero of the zero polynomial is

(a) 0 (b) 1

(c) Any real number (d) Not defined

Sol. The zero (or degree) of the zero polynomial is undefined.
Hence, (d) is the correct answer.

9. Zero of the polynomial $p(x) = 2x + 5$ is

(a) $-\frac{2}{5}$ (b) $-\frac{5}{2}$ (c) $\frac{2}{5}$ (d) $\frac{5}{2}$

Sol. Finding a zero of $p(x)$ is the same as solving an equation $p(x) = 0$.
Now, $p(x) = 0 \Rightarrow 2x + 5 = 0$,

which gives us $x = -\frac{5}{2}$.

Therefore, $-\frac{5}{2}$ is the zero of the polynomial.

Hence, (*b*) is the correct answer.

10. One of the zeroes of the polynomial $2x^2 + 7x - 4$ is

(*a*) 2 (*b*) $\frac{1}{2}$ (*c*) $-\frac{1}{2}$ (*d*) –2

Sol. We have $p(x) = 2x^2 + 7x + 4$

(*a*) $p(2) = 2(2)^2 + 7(2) - 4$

$= 8 + 14 - 4$

$= 18 \neq 0$

(*b*) $p\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^2 + 7\left(\frac{1}{2}\right) - 4$

$= 2 \times \frac{1}{4} + \frac{7}{2} - 4 = \frac{1}{2} + \frac{7}{2} - 4 = 4 - 4 = 0$

(*c*) $p\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^2 + 7\left(-\frac{1}{2}\right) - 4$

$= 2 \times \frac{1}{4} - \frac{7}{2} - 4 = \frac{1}{2} - \frac{7}{2} - 4$

$= -3 - 4$

$= -7 \neq 0$

(*d*) $p(-2) = 2(-2)^2 + 7(-2) - 4$

$= 8 - 14 - 4 = -10 \neq 0$

As $p\left(\frac{1}{2}\right) = 0$, we say that $\frac{1}{2}$ is a zero of the polynomial. Hence, $\frac{1}{2}$ is one of the zeroes of the polynomial $2x^2 + 7x - 4$.

Hence, (*b*) is the correct answer.

11. If $x^{51} + 51$ is divided by $x + 1$, the remainder is

(*a*) 0 (*b*) 1 (*c*) 49 (*d*) 50

Sol. If $p(x)$ is divided by $x + a$, then the remainder is $p(-a)$.

Here $p(x) = x^{51} + 51$ is divided by $x + 1$, then

Remainder $= p(-1) = (-1)^{51} + 51 = -1 + 51 = 50$

Hence, (*d*) is the correct answer.

12. If $x + 1$ is a factor of the polynomial $2x^2 + kx$, then the value of k is

(*a*) – 3 (*b*) 4 (*c*) 2 (*d*) –2

Sol. Let $p(x) = 2x^2 + kx$

If $x + 1$ is a factor of $p(x)$, then by factor theorem $p(-1) = 0$

Now, $p(-1) = 0 \Rightarrow 2(-1)^2 + k(-1) = 0$

$\Rightarrow$ $2 - k = 0;\ k = 2$

Hence, (*c*) is the correct answer.

13. $x + 1$ is a factor of the polynomial

(*a*) $x^3 + x^2 - x + 1$ (*b*) $x^3 + x^2 + x + 1$

(*c*) $x^4 + x^3 + x^2 + 1$ (*d*) $x^4 + 3x^3 + 3x^2 + x + 1$

Sol. If $x + 1$ is a factor of $p(x)$, then $p(-1) = 0$

(*a*) Let $p(x) = x^3 + x^2 - x + 1$

$\therefore \quad p(-1) = (-1)^3 + (-1)^2 - (-1) + 1$

$= -1 + 1 + 1 + 1 = 2 \neq 0$

So, $x + 1$ is not a factor of $p(x)$.

(*b*) Let $p(x) = x^3 + x^2 + x + 1$

$\therefore \quad p(-1) = (-1)^3 + (-1)^2 + (-1) + 1$

$= -1 + 1 - 1 + 1 = 0$

(*c*) Let $p(x) = x^4 + x^3 + x^2 + 1$

$\therefore \quad p(-1) = (-1)^4 + (-1)^3 + (-1)^2 + 1$

$= 1 - 1 + 1 + 1 = 2 \neq 0$

(*d*) Let $p(x) = x^4 + 3x^3 + 3x^2 + x + 1$

$\therefore \quad p(-1) = (-1)^4 + 3(-1)^3 + 3(-1)^2 + (-1) + 1$

$= 1 - 3 + 3 - 1 + 1 = 1 \neq 0$

Hence, $x + 1$ is a factor of $x^3 + x^2 + x + 1$.

So, (*b*) is the correct answer.

14. One of the factors of $(25x^2 - 1) + (1 + 5x)^2$ is

(*a*) $5 + x$ (*b*) $5 - x$ (*c*) $5x - 1$ (*d*) $10x$

Sol. $(25x^2 - 1) + (1 + 5x)^2 = (5x)^2 - 1^2 + (5x + 1)^2$

$= (5x - 1)(5x + 1) + (5x + 1)^2 = (5x + 1)(5x - 1 + 5x + 1)$

$= (5x + 1)(10x) = 10x\,(5x + 1)$

Hence, one of the factors of $(25x^2 - 1) + (1 + 5x)^2$ is $10x$. Therefore, (*d*) is the correct answer.

15. The value of $249^2 - 248^2$ is

(*a*) 1^2 (*b*) 477 (*c*) 487 (*d*) 497

Sol. $(249)^2 - (248)^2 = (249 + 248)(249 - 248)$

$= (497)(1) = 497$

Hence, (*d*) is the correct answer.

16. The factorisation of $4x^2 + 8x + 3$ is

(*a*) $(x + 1)(x + 3)$ (*b*) $(2x + 1)(2x + 3)$

(*c*) $(2x + 2)(2x + 5)$ (*d*) $(2x - 1)(2x - 3)$

Sol. $4x^2 + 8x + 3 = 4x^2 + 6x + 2x + 3$

$= 2x\,(2x + 3) + 1\,(2x + 3) = (2x + 1)(2x + 3)$

Hence, (*b*) is the correct answer.

17. Which of the following is a factor of $(x + y)^3 - (x^3 + y^3)$?

(*a*) $x^2 + y^2 + 2xy$ (*b*) $x^2 + y^2 - xy$

(*c*) xy^2 (*d*) $3xy$

Sol. $(x + y)^3 - (x^3 + y^3) = x^3 + y^3 + 3xy\,(x + y) - x^3 - y^3$

$= 3xy(x + y)$

So, $3xy$ is a factor of $(x + y)^3 – (x^3 + y^3)$.
Hence, (*d*) is the correct answer.

18. The coefficient of x in the expansion of $(x + 3)^3$ is

(*a*) 1 (*b*) 9 (*c*) 18 (*d*) 27

Sol. Using $(a + b)^3 = a^3 + b^3 + 3ab\,(a + b)$, we get

$$(x+3)^3 = x^3 + 3^3 + 3 \times x \times 3\,(x+3)$$
$$= x^3 + 27 + 9x^2 \times 27x$$

Therefore, the coefficient of x is 27.
Hence, (*d*) is the correct answer.

19. If $\frac{x}{y} + \frac{y}{x} = -1$, the value of $x^3 – y^3$ is

(*a*) 1 (*b*) –1 (*c*) 0 (*d*) $\frac{1}{2}$

Sol. $$\frac{x}{y} + \frac{y}{x} = -1 \quad \Rightarrow \quad \frac{x^2 + y^2}{xy} = -1$$

$\Rightarrow$ $x^2 + y^2 = –xy$

Now, $x^3 – y^3 = (x – y)\,(x^2 + y^2 + xy)$

$= (x – y)\,(–xy + xy)$ $[\because\ x^2 + y^2 = – xy]$

$= (x – y)\,(0)$

$= 0$

Hence, (*c*) is the correct answer.

20. If $49x^2 – b = \left(7x + \frac{1}{2}\right)\left(7x - \frac{1}{2}\right)$, then the value of b is

(*a*) 0 (*b*) $\frac{1}{\sqrt{2}}$ (*c*) $\frac{1}{4}$ (*d*) $\frac{1}{2}$

Sol. $$49x^2 – b = \left(7x + \frac{1}{2}\right)\left(7x - \frac{1}{2}\right)$$

$\Rightarrow$ $$49x^2 – b = (7x)^2 – \left(\frac{1}{2}\right)^2$$

$$= 49x^2 – \frac{1}{4} \quad [\because\ (a + b)\,(a – b) = a^2 – b^2]$$

So, we get $b = \frac{1}{4}$.
Hence, (*c*) is the correct answer.

21. If $a + b + c = 0$, then the value of $a^3 + b^3 + c^3$ is equal to

(*a*) 0 (*b*) abc (*c*) $3abc$ (*d*) $2abc$

Sol. We know that

$a^3 + b^3 + c^3 – 3abc = (a + b + c)\,(a^2 + b^2 + c^2 – ab – bc – ca)$

As $a + b + c = 0$, so, $a^3 + b^3 + c^3 – 3abc = (0)\,(a^2 + b^2 + c^2 – ab – bc – ca) = 0$

Hence, $a^3 + b^3 + c^3 = 3abc$.

Therefore, (*c*) $3abc$ is the correct answer.

EXERCISE 2.2

1. Which of the following expressions are polynomials? Justify your answer.

(*i*) 8 (*ii*) $\sqrt{3}x^2 - 2x$ (*iii*) $1 - \sqrt{5x}$

(*iv*) $\frac{1}{5x^{-2}} + 5x + 7$ (*v*) $\frac{(x-2)(x-4)}{x}$ (*vi*) $\frac{1}{x+1}$

(*vii*) $\frac{1}{7}a^3 - \frac{2}{\sqrt{3}}a^2 + 4a - 7$ (*viii*) $\frac{1}{2x}$

Sol. (*i*) 8 is a constant polynomial.

(*ii*) $\sqrt{3}x^2 - 2x$

In each term of this expression, the exponent of the variable x is a whole number. Hence, it is a polynomial.

(*iii*) $1 - \sqrt{5x} = 1 - \sqrt{5}x^{\frac{1}{2}}$

Here, the exponent of the second term, *i.e.,* $x^{\frac{1}{2}}$ is $\frac{1}{2}$, which is not a whole number. Hence, the given algebraic expression is not a polynomial.

(*iv*) $\frac{1}{5x^{-2}} + 5x + 7 = \frac{1}{5}x^2 + 5x + 7$

In each term of this expression, the exponent of the variable x is a whole number. Hence, it is a polynomial.

(*v*) $\frac{(x-2)(x-4)}{x} = \frac{x^2 - 6x + 8}{x} = x - 6 + \frac{8}{x} = x - 6 + 8x^{-1}$

Here, the exponent of variable x in the third term, *i.e.,* in $8x^{-1}$, is -1, which is not a whole number. So, this algebraic expression is not a polynomial.

(*vi*) $\frac{1}{x+1} = (x+1)^{-1}$ which cannot be reduced to an expression in which the exponent of the variable x have only whole numbers in each of its terms. So, this algebraic expression is not a polynomial.

(*vii*) $\frac{1}{7}a^3 - \frac{2}{\sqrt{3}}a^2 + 4a - 7$

In this expression, the exponent of a in each term is a whole number, so this expression is a polynomial.

(*viii*) $\frac{1}{2x} = \frac{1}{2}x^{-1}$

Here, the exponent of the variable x is -1, which is not a whole number. So, this algebraic expression is not a polynomial.

2. Write whether the following statements are true or false. Justify your answer.

(*i*) A binomial can have atmost two terms.

(*ii*) Every polynomial is a binomial.

(*iii*) A binomial may have degree 5.

(*iv*) Zero of a polynomial is always 0.

(*v*) A polynomial cannot have more than one zero.

(*vi*) The degree of the sum of two polynomials each of degree 5 is always 5.

Sol. (*i*) The given statement is false because binomial have exactly two terms.

(*ii*) A polynomial can be a monomial, binomial trinomial or can have finite number of terms. For example, $x^4 + x^3 + x^2 + 1$ is a polynomial but not binomial.
Hence, the given statement is false.

(*iii*) The given statement is true because a binomial is a polynomial whose degree is a whole number ≥ 1. For example, $x^5 - 1$ is a binomial of degree 5.

(*iv*) The given statement is false, because zero of polynomial can be any real number.

(*v*) The given statement is false, because a polynomial can have any number of zeroes which depends on the degree of the polynomial.

(*vi*) The given statement is false. For example, consider the two polynomials $-x^5 + 3x^2 + 4$ and $x^5 + x^4 + 2x^3 + 3$. The degree of each of these polynomials is 5. Their sum is $x^4 + 2x^3 + 3x^2 + 7$. The degree of this polynomial is not 5.

EXERCISE 2.3

1. Classify the following polynomials as polynomials in one variable, two variables etc.

(*i*) $x^2 + x + 1$ (*ii*) $y^3 - 5y$

(*iii*) $xy + yz + zx$ (*iv*) $x^2 - 2xy + y^2 + 1$

Sol. (*i*) $x^2 + x + 1$ is a polynomial in one variable.

(*ii*) $y^3 - 5y$ is a polynomial in one variable.

(*iii*) $xy + yz + zx$ is a polynomial in three variables.

(*iv*) $x^2 - 2xy + y^2 + 1$ is a polynomial in two variables.

2. Determine the degree of each of the following polynomials:

(*i*) $2x - 1$ (*ii*) -10 (*iii*) $x^3 - 9x + 3x^5$ (*iv*) $y^3(1 - y^4)$

Sol. (*i*) Since the highest power of x is 1, the degree of the polynomial $2x - 1$ is 1.

(*ii*) -10 is a non-zero constant. A non-zero constant term is always regarded as having degree 0.

(*iii*) Since the highest power of x is 5, the degree of the polynomial $x^3 - 9x + 3x^5$ is 5.

(*iv*)
$$y^3(1 - y^4) = y^3 - y^7$$

Since the highest power of y is 7, the degree of the polynomial is 7.

3. For the polynomial $\frac{x^3+2x+1}{5} - \frac{7}{2}x^2 - x^6$, write

(*i*) the degree of the polynomial.

(*ii*) the coefficient of x^3.

(*iii*) the coefficient of x^6.

(*iv*) the constant term.

Sol. (*i*) We know that highest power of variable in a polynomial is the degree of the polynomial.
In the given polynomial, the term with highest of x is $-x^6$ and the exponent of x in this term in 6.

(*ii*) The coefficient of x^3 is $\frac{1}{5}$. (*iii*) The coefficient of x^6 is -1.

(*iv*) The constant term is $\frac{1}{5}$.

4. Write the coefficient of x^2 in each of the following:

(*i*) $\frac{\pi}{6}x + x^2 - 1$ (*ii*) $3x - 5$

(*iii*) $(x-1)(3x-4)$ (*iv*) $(2x-5)(2x^2-3x+1)$

Sol. (*i*) The coefficient of x^2 in the given polynomial is 1.

(*ii*) The given polynomial can be written as 0. $x^2 + 3x - 5$. So, the coefficient of x^2 in the given polynomial is 0.

(*iii*) The given polynomial can be written as:

$$\begin{aligned}(x-1)(3x-4) &= 3x^2 - 4x - 3x + 4\\ &= 3x^2 - 7x + 4\end{aligned}$$

So, the coefficient of x^2 in the given polynomial is 3.

(*iv*) The given polynomial can be written as:

$$\begin{aligned}(2x-5)(2x^2-3x+1) &= 4x^3 - 6x^2 + 2x - 10x^2 + 15x - 5\\ &= 4x^3 - 16x^2 + 17x - 5\end{aligned}$$

So, the coefficient of x^2 in the given polynomial is -16.

5. Classify the following as a constant, linear, quadratic and cubic polynomials:

(*i*) $2 - x^2 + x^3$ (*ii*) $3x^3$ (*iii*) $5t - \sqrt{7}$

(*iv*) $4 - 5y^2$ (*v*) 3 (*vi*) $2 + x$

(*vii*) $y^3 - y$ (*viii*) $1 + x + x^2$ (*ix*) t^2 (*x*) $\sqrt{2}x - 1$

EXERCISE 2.3

1. Classify the following polynomials as polynomials in one variable, two variables etc.

(*i*) x^2+x+1 (*ii*) y^3-5y

(*iii*) $xy+yz+zx$ (*iv*) $x^2-2xy+y^2+1$

Sol. (*i*) x^2+x+1 is a polynomial in one variable.

(*ii*) y^3-5y is a polynomial in one variable.

(*iii*) $xy+yz+zx$ is a polynomial in three variables.

(*iv*) $x^2-2xy+y^2+1$ is a polynomial in two variables.

2. Determine the degree of each of the following polynomials:

(*i*) $2x-1$ (*ii*) -10 (*iii*) $x^3-9x+3x^5$ (*iv*) $y^3(1-y^4)$

Sol. (*i*) Since the highest power of x is 1, the degree of the polynomial $2x-1$ is 1.

(*ii*) -10 is a non-zero constant. A non-zero constant term is always regarded as having degree 0.

(*iii*) Since the highest power of x is 5, the degree of the polynomial $x^3-9x+3x^5$ is 5.

(*iv*) $$y^3(1-y^4)=y^3-y^7$$

Since the highest power of y is 7, the degree of the polynomial is 7.

$$= (4-8+3)-(1+4+3)+\left(\frac{1}{4}-2+3\right)$$

$$= -1-8+\frac{5}{4}$$

$$= -9+\frac{5}{4} = \frac{-36+5}{4} = \frac{-31}{4}$$

9. Find $p(0), p(1), p(-2)$ for the following polynomials:

(*i*) $p(x) = 10x - 4x^2 - 3$ (*ii*) $p(y) = (y+2)(y-2)$

Sol. (*i*) We have $p(x) = 10x - 4x^2 - 3$

$\therefore$ $p(0) = 10(0) - 4(0)^2 - 3$

$= 0 - 0 - 3 = -3$

And, $p(1) = 10(1) - 4(1)^2 - 3$

$= 10 - 4 - 3 = 10 - 7 = 3$

And, $p(-2) = 10(-2) - 4(-2)^2 - 3$

$= -20 - 4(4) - 3 = -20 - 16 - 3 = -39$

(*ii*) We have $p(y) = (y+2)(y-2) = y^2 - 4$

$\therefore$ $p(0) = (0)^2 - 4$

$= 0 - 4 = -4$

And, $p(1) = (1)^2 - 4$

$= 1 - 4 = -3$

And, $p(-2) = (-2)^2 - 4$

$= 4 - 4 = 0$

10. Verify whether the following are **True** or **False**.

(*i*) -3 is a zero of $x - 3$. (*ii*) $-\frac{1}{3}$ is a zero of $3x + 1$.

(*iii*) $\frac{-4}{5}$ is a zero of $4 - 5y$. (*iv*) 0 and 2 are the zeros of $t^2 - 2t$.

(*v*) -3 is a zero of $y^2 + y - 6$.

Sol. A zero of a polynomial $p(x)$ is a number c such that $p(c) = 0$

(*i*) Let $p(x) = x - 3$

$\therefore$ $p(-3) = -3 - 3 = -6 \neq 0$

Hence, -3 is not a zero of $x - 3$.

(*ii*) Let $p(x) = 3x + 1$

$$\therefore \quad p\left(-\frac{1}{3}\right) = 3\left(-\frac{1}{3}\right) + 1 = -1 + 1 = 0$$

Hence, $-\frac{1}{3}$ is a zero of $p(x) = 3x + 1$.

(*iii*) Let $p(y) = 4 - 5y$

$\therefore \quad p\left(-\frac{4}{5}\right) = 4 - 5\left(\frac{-4}{5}\right) = 4 + 4 = 8 \neq 0$

Hence, $-\frac{4}{5}$ is not a zero of $4 - 5y$.

(*iv*) Let $p(t) = t^2 - 2t$

$\therefore \quad p(0) = (0)^2 - 2(0) = 0$

and $\quad p(2) = (2)^2 - 2(2) = 4 - 4 = 0$

Hence, 0 and 2 are zeroes of the polynomial $p(t) = t^2 - 2t$.

(*v*) Let $p(y) = y^2 + y - 6$

$\therefore \quad p(-3) = (-3)^2 + (-3) - 6 = 9 - 3 - 6 = 0$

Hence, -3 is a zero of the polynomial $y^2 + y - 6$.

11. Find the zeroes of the polynomial in each of the following:

(*i*) $p(x) = x - 4$ (*ii*) $g(x) = 3 - 6x$

(*iii*) $q(x) = 2x - 7$ (*iv*) $h(y) = 2y$

Sol. (*i*) Solving the equation $p(x) = 0$, we get

$x - 4 = 0$, which gives us $x = 4$

So, 4 is a zero of the polynomial $x - 4$.

(*ii*) Solving the equation $g(x) = 0$, we get

$3 - 6x = 0$, which gives us $x = \frac{1}{2}$

So, $\frac{1}{2}$ is a zero of the polynomial $3 - 6x$.

(*iii*) Solving the equation $q(x) = 0$, we get

$2x - 7 = 0$, which gives us $x = \frac{7}{2}$

So, $\frac{7}{2}$ is a zero of the polynomial $2x - 7$.

(*iv*) Solving the equation $h(y) = 0$, we get

$2y = 0$, which gives us $y = 0$

So, 0 is a zero of the polynomial $2y$.

12. Find the zeroes of the polynomial $(x - 2)^2 - (x + 2)^2$.

Sol. Let $p(x) = (x - 2)^2 - (x + 2)^2$

As finding a zero of $p(x)$, is same as solving the equation $p(x) = 0$

So, $p(x) = 0 \Rightarrow (x - 2)^2 - (x + 2)^2 = 0$

$\Rightarrow (x - 2 + x + 2)(x - 2 - x - 2) = 0$

$\Rightarrow 2x(-4) = 0 \Rightarrow -8x = 0 \Rightarrow x = 0$

Hence, $x = 0$ is the only one zero of $p(x)$.

13. By actual division, find the quotient and the remainder when the first polynomial is divided by the second polynomial: $x^4 + 1; x + 1$.

Sol. By actual division, we have

$$
\begin{array}{r|l}
 & x^3 + x^2 + x + 1 \\ \hline
x - 1 & x^4 + 1 \\
 & \underset{-}{x^4} \underset{+}{-} x^3 \\ \hline
 & x^3 + 1 \\
 & \underset{-}{x^3} \underset{+}{-} x^2 \\ \hline
 & x^2 + 1 \\
 & \underset{-}{x^2} \underset{+}{-} x \\ \hline
 & x + 1 \\
 & \underset{-}{x} \underset{+}{-} 1 \\ \hline
 & \quad 2
\end{array}
$$

14. By Remainder Theorem find the remainder, when $p(x)$ is divided by $g(x)$, where

(*i*) $p(x) = x^3 - 2x^2 - 4x - 1, \quad g(x) = x + 1$

(*ii*) $p(x) = x^3 - 3x^2 + 4x + 50, \quad g(x) = x - 3$

(*iii*) $p(x) = 4x^3 - 12x^2 + 14x - 3, g(x) = 2x - 1$

(*iv*) $p(x) = x^3 - 6x^2 + 2x - 4, g(x) = 1 - \frac{3}{2}x$

Sol. (*i*) We have $g(x) = x + 1$

$\Rightarrow \quad x + 1 = 0$

$\Rightarrow \quad x = -0$

$$
\begin{aligned}
\text{Remainder} &= p(-1) \\
&= (-1)^3 - 2(-1)^2 - 4(-1) - 1 = -1 - 2 + 4 - 1 \\
&= 0
\end{aligned}
$$

(*ii*) We have $g(x) = x - 3$

$\Rightarrow \quad x - 3 = 0$

$\Rightarrow \quad x = 3$

$$
\begin{aligned}
\text{Remainder} &= p(3) \\
&= (3)^3 - 3(3)^2 + 4(3) + 50 = 27 - 27 + 12 + 50 \\
&= 62
\end{aligned}
$$

(*iii*) We have $g(x) = 2x - 1$

$\Rightarrow \quad 2x - 1 = 0$

$\Rightarrow \quad 2x = 1 \Rightarrow x = \frac{1}{2}$

$$\text{Remainder} = p\left(\frac{1}{2}\right)$$

$$= 4\left(\frac{1}{2}\right)^3 - 12\left(\frac{1}{2}\right)^2 + 14\left(\frac{1}{2}\right) - 3$$

$$= 4\left(\frac{1}{8}\right) - 12\left(\frac{1}{4}\right) + 7 - 3$$

$$= \frac{1}{2} - 3 + 7 - 6 = \frac{1}{2} - 2 = \frac{-3}{2}$$

(*iv*) $g(x) = 0 \quad \Rightarrow 1 - \frac{3}{2}x = 0;\ x = \frac{2}{3}$

$$\text{Remainder} = p\left(\frac{2}{3}\right) = \frac{8}{27} - \frac{24}{9} + \frac{4}{3} - 4$$

$$= \frac{8 - 72 + 36 - 108}{27} = \frac{-136}{27}$$

15. Check whether $p(x)$ is a multiple of $g(x)$ or not:

(*i*) $p(x) = x^3 - 5x^2 + 4x - 3,\ g(x) = x - 2$

(*ii*) $p(x) = 2x^3 - 11x^2 - 4x + 5,\ g(x) = 2x + 1$

Sol. (*i*) $p(x)$ will be a multiple $g(x)$ if $g(x)$ divides $p(x)$.

Now, $g(x) = x - 2$ gives $x = 2$

$$\text{Remainder} = p(2) = (2)^3 - 5(2)^2 + 4(2) - 3$$

$$= 8 - 5(4) + 8 - 3 = 8 - 20 + 8 - 3$$

$$= -7$$

Since remainder $\neq 0$, so $p(x)$ is not a multiple of $g(x)$.

(*ii*) $p(x)$ will be a multiple of $g(x)$ if $g(x)$ divides $p(x)$.

Now, $g(x) = 2x + 1$ give $x = -\frac{1}{2}$

$$\text{Remainder} = p\left(\frac{-1}{2}\right) = 2\left(\frac{-1}{2}\right)^3 - 11\left(\frac{-1}{2}\right)^2 - 4\left(\frac{-1}{2}\right) + 5$$

$$= 2\left(\frac{-1}{8}\right) - 11\left(\frac{1}{4}\right) + 2 + 5 = \frac{-1}{4} - \frac{11}{4} + 7$$

$$= \frac{-1 - 11 + 28}{4} = \frac{16}{4} = 4$$

Since remainder $\neq 0$, so $p(x)$ is not a multiple of $g(x)$.

16. Show that:

(i) $x + 3$ is a factor of $69 + 11x - x^2 + x^3$.

(ii) $2x - 3$ is a factor of $x + 2x^3 - 9x^2 + 12$.

Sol. (i) Let $p(x) = 69 + 11x - x^2 + x^3$, $g(x) = x + 3$.

$$g(x) = x + 3 = 0 \text{ gives } x = -3$$

$g(x)$ will be a factor of $p(x)$ if $p(-3) = 0$ (Factor theorem)

Now,
$$\begin{aligned} p(-3) &= 69 + 11(-3) - (-3)^2 + (-3)^3 \\ &= 69 - 33 - 9 - 27 \\ &= 0 \end{aligned}$$

Since, $p(-3) = 0$, so, $g(x)$ is a factor of $p(x)$.

(ii) Let $p(x) = x + 2x^3 - 9x^2 + 12$ and $g(x) = 2x - 3$

$$g(x) = 2x - 3 = 0 \text{ gives } x = \frac{3}{2}$$

$g(x)$ will be a factor of $p(x)$ if $p\left(\frac{3}{2}\right) = 0$ (Factor theorem)

Now,
$$p\left(\frac{3}{2}\right) = \frac{3}{2} + 2\left(\frac{3}{2}\right)^3 - 9\left(\frac{3}{2}\right)^2 + 12 = \frac{3}{2} + 2\left(\frac{27}{8}\right) - 9\left(\frac{9}{4}\right) + 12$$
$$= \frac{3}{2} + \frac{27}{4} - \frac{81}{4} + 12 = \frac{6 + 27 - 81 + 48}{4} = \frac{0}{4} = 0$$

Since, $p\left(\frac{3}{2}\right) = 0$, so, $g(x)$ is a factor of $p(x)$.

17. Determine which of the following polynomials has $x - 2$ a factor:

(i) $3x^2 + 6x - 24$ (ii) $4x^2 + x - 2$

Sol. We knwo that if $(x - a)$ is a factor of $p(x)$, then $p(a) = 0$.

(i) Let $p(x) = 3x^2 + 6x - 24$

If $x - 2$ is a factor of $p(x) = 3x^2 + 6x - 24$, then $p(2)$ should be equal to 0.

Now,
$$\begin{aligned} p(2) &= 3(2)^2 + 6(2) - 24 \\ &= 3(4) + 6(2) - 24 \\ &= 12 + 12 - 24 \\ &= 0 \end{aligned}$$

$\therefore$ By factor theorem, $(x - 2)$ is a factor of $3x^2 + 6x - 24$.

(ii) Let $p(x) = 4x^2 + x - 2$.

If $x - 2$ is a factor of $p(x) = 4x^2 + x - 2$, then, $p(2)$ should be equal to 0.

Now,
$$p(2) = 4(2)^2 + 2 - 2$$

$$= 4(4) + 2 - 2$$
$$= 16 + 2 - 2$$
$$= 16 \neq 0$$

$\therefore$ $x - 2$ is not a factor of $4x^2 + x - 2$.

18. Show that $p - 1$ is a factor of $p^{10} - 1$ and also of $p^{11} - 1$.

Sol. If $p - 1$ is a factor of $p^{10} - 1$, then $(1)^{10} - 1$ should be equal to zero.

Now, $(1)^{10} - 1 = 1 - 1 = 0$

Therefore, $p - 1$ is a factor of $p^{10} - 1$.

Again, if $p - 1$ is a factor of $p^{11} - 1$, then $(1)^{11} - 1$ should be equal to zero.

Now, $(1)^{11} - 1 = 1 - 1 = 0$

Therefore, $p - 1$ is a factor of $p^{11} - 1$.

Hence, $p - 1$ is a factor of $p^{10} - 1$ and also of $p^{11} - 1$.

19. For what value of m is $x^3 - 2mx^2 + 16$ divisible by $x + 2$?

Sol. If $x^3 - 2mx^2 + 16$ is divisible by $x + 2$, then $x + 2$ is a factor of $x^3 - 2mx^2 + 16$.

Now, let $p(x) = x^3 - 2mx^2 + 16$.

As $x + 2 = x - (-2)$ is a factor of $x^3 - 2mx^2 + 16$

so $p(-2) = 0$

Now, $p(-2) = (-2)^3 - 2m(-2)^2 + 16$

$= -8 - 8m + 16 = 8 - 8m$

Now, $p(-2) = 0$

$\Rightarrow$ $8 - 8m = 0$

$\Rightarrow$ $m = 8 \div 8$

$\Rightarrow$ $m = 1$

Hence, for $m = 1$, $x + 2$ is a factor of $x^3 - 2mx^2 + 16$, so $x^3 - 2mx^2 + 16$ is completely divisible by $x + 2$.

20. If $x + 2a$ is a factor of $x^5 - 4a^2x^3 + 2x + 2a + 3$, find a.

Sol. Let $p(x) = x^5 - 4a^2x^3 + 2x + 2a + 3$

If $x - (-2a)$ is a factor of $p(x)$, then $p(-2a) = 0$

$\therefore$ $p(-2a) = (-2a)^5 - 4a^2(-2a)^3 + 2(-2a) + 2a + 3$

$= -32a^5 + 32a^5 - 4a + 2a + 3$

$= -2a + 3$

Now, $p(-2a) = 0$

$\Rightarrow$ $-2a + 3 = 0$

$\Rightarrow$ $a = \dfrac{3}{2}$

21. Find the value of m so that $2x - 1$ be a factor of $8x^4 + 4x^3 - 16x^2 + 10x + m$.

Sol. Let $p(x) = 8x^4 + 4x^3 - 16x^2 + 10x + m$.

As $(2x - 1)$ is a factor of $p(x)$

$\therefore \quad p\left(\frac{1}{2}\right) = 0$ [By factor theorem]

$\Rightarrow \quad 8\left(\frac{1}{2}\right)^4 + 4\left(\frac{1}{2}\right)^3 - 16\left(\frac{1}{2}\right)^2 + 10\left(\frac{1}{2}\right) + m = 0$

$\Rightarrow \quad 8\left(\frac{1}{16}\right) + 4\left(\frac{1}{8}\right) - 16\left(\frac{1}{4}\right) + 5 + m = 0$

$\Rightarrow \quad \frac{1}{2} + \frac{1}{2} - 4 + 5 + m = 0$

$\Rightarrow \quad 2 + m = 0 \quad \Rightarrow \quad m = -2$

22. If $x + 1$ is a factor of $ax^3 + x^2 - 2x + 4a - 9$, find the value of a.

Sol. Let $p(x) = ax^3 + x^2 - 2x + 4a - 9$.

As $(x + 1)$ is a factor of $p(x)$

$\therefore \quad p(-1) = 0$ [By factor theorem]

$\Rightarrow \quad a(-1)^3 + (-1)^2 - 2(-1) + 4a - 9 = 0$

$\Rightarrow \quad a(-1) + 1 + 2 + 4a - 9 = 0$

$\Rightarrow \quad -a + 4a - 6 = 0$

$\Rightarrow \quad 3a - 6 = 0 \Rightarrow 3a = 6 \Rightarrow a = 2$

23. Factorise:

(*i*) $x^2 + 9x + 18$ (*ii*) $6x^2 + 7x - 3$

(*iii*) $2x^2 - 7x - 15$ (*iv*) $84 - 2r - 2r^2$

Sol. (*i*) In order to factorise $x^2 + 9x + 18$, we have to find two numbers p and q such that $p + q = 9$ and $pq = 18$.

Clearly, $6 + 3 = 9$ and $6 \times 3 = 18$.

So, we write the middle term $9x$ as $6x + 3x$.

$$\begin{aligned}\therefore \quad x^2 + 9x + 18 &= x^2 + 6x + 3x + 18\\ &= x(x + 6) + 3(x + 6)\\ &= (x + 6)(x + 3)\end{aligned}$$

(*ii*) In order to factorise $6x^2 + 7x - 3$, we have to find two numbers p and q such that $p + q = 7$ and $pq = -18$.

Clearly, $9 + (-2) = 7$ and $9 \times (-2) = -18$.

So, we write the middle term $7x$ as $9x + (-2x)$, i.e., $9x - 2x$.

$$\therefore \quad 6x^2 + 7x - 3 = 6x^2 + 9x - 2x - 3$$
$$= 3x(2x+3) - 1(2x+3)$$
$$= (2x+3)(3x-1)$$

(*iii*) In order to factorise $2x^2 - 7x - 15$, we have to find two numbers p and q such that $p + q = -7$ and $pq = -30$.

Clearly, $(-10) + 3 = -7$ and $(-10) \times 3 = -30$.

So, we write the middle term $-7x$ as $(-10x) + 3x$.

$$\therefore \quad 2x^2 - 7x - 15 = 2x^2 - 10x + 3x - 15$$
$$= 2x(x-5) + 3(x-5)$$
$$= (x-5)(2x+3)$$

(*iv*) In order to factorise $84 - 2r - 2r^2$, we have to find two numbers p and q such that $p + q = -2$ and $pq = -168$.

Clearly, $(-14) + 12 = -2$ and $(-14) \times 12 = -168$.

So, we write the middle term $-2r$ as $(-14r) + 12r$.

$$\therefore \quad 84 - 2r - 2r^2 = -2r^2 - 2r + 84$$
$$= -2r^2 - 14r + 12r + 84$$
$$= -2r(r+7) + 12(r+7)$$
$$= (r+7)(-2r+12)$$
$$= -2(r+7)(r-6) = -2(r-6)(r+7)$$

24. Factorise:

(*i*) $2x^3 - 3x^2 - 17x + 30$ (*ii*) $x^3 - 6x^2 + 11x - 6$

(*iii*) $x^3 + x^2 - 4x - 4$ (*iv*) $3x^3 - x^2 - 3x + 1$

Sol. (*i*) Let $f(x) = 2x^3 - 3x^2 - 17x + 30$ be the given polynomial. The factors of the constant term +30 are $\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15, \pm 30$. The factor of coefficient of x^3 is 2. Hence, possible rational roots of $f(x)$ are:

$$\pm 1, \pm 3, \pm 5, \pm 15, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{15}{2}.$$

We have $\quad f(2) = 2(2)^3 - 3(2)^2 - 17(2) + 30$
$$= 2(8) - 3(4) - 17(2) + 30$$
$$= 16 - 12 - 34 + 30 = 0$$

and $\quad f(-3) = 2(-3)^3 - 3(-3)^2 - 17(-3) + 30$
$$= 2(-27) - 3(9) - 17(-3) + 30$$
$$= -54 - 27 + 51 + 30 = 0$$

So, $(x-2)$ and $(x+3)$ are factors of $f(x)$.

$\Rightarrow \quad x^2 + x - 6$ is a factor of $f(x)$.

Let us now divide $f(x) = 2x^3 - 3x^2 - 17x + 30$ by $x^2 + x - 6$ to get the other factors of $f(x)$.

By long division, we have

$$
\begin{array}{r|l|l}
x^2 + x - 6 & 2x^3 - 3x^2 - 17x + 30 & 2x - 5 \\
 & 2x^3 + 2x^2 - 12x & \\
 & \underline{-\quad\ -\qquad +\qquad\qquad} & \\
 & -5x^2 - 5x + 30 & \\
 & -5x^2 - 5x + 30 & \\
 & \underline{+\qquad +\qquad -\qquad} & \\
 & \qquad 0 &
\end{array}
$$

$\therefore \quad 2x^3 - 3x^2 - 17x + 30 = (x^2 + x - 6)(2x - 5)$

$\Rightarrow \quad 2x^3 - 3x^2 - 17x + 30 = (x - 2)(x + 3)(2x - 5)$

Hence, $2x^3 - 3x^2 - 17x + 30 = (x - 2)(x + 3)(2x - 5)$

(*ii*) Let $f(x) = x^3 - 6x^2 + 11x - 6$ be the given polynomial. The factors of the constant term -6 are $\pm 1, \pm 2, \pm 3$ and ± 6.

We have, $\quad f(1) = (1)^3 - 6(1)^2 + 11(1) - 6 = 1 - 6 + 11 - 6 = 0$

and, $\quad f(2) = (2)^3 - 6(2)^2 + 11(2) - 6 = 8 - 24 + 22 - 6 = 0$

So, $(x - 1)$ and $(x - 2)$ are factors of $f(x)$.

$\Rightarrow \quad (x - 1)(x - 2)$ is also a factor of $f(x)$.

$\Rightarrow \quad x^2 - 3x + 2$ is a factor of $f(x)$.

Let us now divide $f(x) = x^3 - 6x^2 + 11x - 6$ by $x^2 - 3x + 2$ to get the other factors of $f(x)$.

By long division, we have

$$
\begin{array}{r|l|l}
x^2 - 3x + 2 & x^3 - 6x^2 + 11x - 6 & x - 3 \\
 & x^3 - 3x^2 + 2x & \\
 & \underline{-\quad +\qquad -\qquad\qquad} & \\
 & -3x^2 + 9x - 6 & \\
 & -3x^2 + 9x - 6 & \\
 & \underline{+\qquad -\qquad +\qquad} & \\
 & \qquad 0 &
\end{array}
$$

$\therefore \quad x^3 - 6x^2 + 11x - 6 = (x^2 - 3x + 2)(x - 3)$

$\Rightarrow \quad x^3 - 6x^2 + 11x - 6 = (x - 1)(x - 2)(x - 3)$

Hence, $x^3 - 6x^2 + 11x - 6 = (x - 1)(x - 2)(x - 3)$

(*iii*) Let $f(x) = x^3 + x^2 - 4x - 4$ be the given polynomial. The factors of the constant term -4 are $\pm 1, \pm 2, \pm 4$.

We have,

$$f(-1) = (-1)^3 + (-1)^2 - 4(-1) - 4 = -1 + 1 + 4 - 4 = 0$$

and, $\quad f(2) = (2)^3 + (2)^2 - 4(2) - 4 = 8 + 4 - 8 - 4 = 0$

So, $(x + 1)$ and $(x - 2)$ are factors of $f(x)$.

$\Rightarrow \quad (x + 1)(x - 2)$ is also a factor of $f(x)$.

$\Rightarrow \quad x^2 - x - 2$ is a factor of $f(x)$.

Let us now divide $f(x) = x^3 + x^2 - 4x - 4$ by $x^2 - x - 2$ to get the other factors of $f(x)$.

By long division, we have

$$\begin{array}{r|l|l}
x^2 - x - 2 & x^3 + x^2 - 4x - 4 & x + 2 \\
 & x^3 - x^2 + 2x & \\
 & \underline{- \quad + \quad +} & \\
 & 2x^2 - 2x - 4 & \\
 & 2x^2 - 2x - 4 & \\
 & \underline{- \quad + \quad +} & \\
 & 0 &
\end{array}$$

$\therefore \quad x^3 + x^2 - 4x - 4 = (x^2 - x - 2)(x + 2)$

$\Rightarrow \quad x^3 + x^2 - 4x - 4 = (x + 1)(x - 2)(x + 2)$

Hence, $x^3 + x^2 - 4x - 4 = (x - 2)(x + 1)(x + 2)$

(*iv*) Let $f(x) = 3x^3 - x^2 - 3x + 1$ be the given polynomial. The factors of the constant term $+1$ are ± 1. The factor of coefficient of x^3 is 3. Hence,

possible rational roots of $f(x)$ are: $\pm \frac{1}{3}$.

We have,

$$f(1) = 3(1)^3 - (1)^2 - 3(1) + 1 = 3 - 1 - 3 + 1 = 0$$

and $\quad f(-1) = 3(-1)^3 - (-1)^2 - 3(-1) + 1 = -3 - 1 + 3 + 1 = 0$

So, $(x - 1)$ and $(x + 1)$ are factors of $f(x)$.

$\Rightarrow \quad (x - 1)(x + 1)$ is also a factor of $f(x)$.

$\Rightarrow \quad x^2 - 1$ is a factor of $f(x)$.

Let us now divide $f(x) = 3x^3 - x^2 - 3x + 1$ by $x^2 - 1$ to get the other factors of $f(x)$.

By long division, we have

$$\begin{array}{r|l|l}
x^2 - 1 & 3x^3 - x^2 - 3x + 1 & 3x - 1 \\
 & 3x^3 \quad\quad - 3x & \\
 & \underline{- \quad\quad\quad +} & \\
 & -x^2 + 1 & \\
 & -x^2 + 1 & \\
 & \underline{+ \quad\; -} & \\
 & 0 &
\end{array}$$

$\therefore \quad 3x^3 - x^2 - 3x + 1 = (x^2 - 1)(3x - 1)$

$\Rightarrow \quad 3x^3 - x^2 - 3x + 1 = (x-1)(x+1)(3x-1)$

Hence, $\quad 3x^3 - x^2 - 3x + 1 = (x-1)(x+1)(3x-1)$

25. Using suitable identity, evaluate the following:

(*i*) 103^3 (*ii*) 101×102 (*iii*) 999^2

Sol. (*i*) $103^3 = (100+3)^3$

Now, using identity $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$, we have

$$\begin{aligned}(100+3)^3 &= (100)^3 + (3)^3 + 3(100)(3)(100+3)\\ &= 1000000 + 27 + 900(100+3)\\ &= 1000000 + 27 + 90000 + 2700\\ &= 1092727\end{aligned}$$

(*ii*) $101 \times 102 = (100+1)(100+2)$

Now, using identity $(x+a)(x+b) = x^2 + (a+b)x + ab$, we have

$$\begin{aligned}(100+1)(100+2) &= (100)^2 + (1+2)100 + (1)(2)\\ &= 10000 + (3)100 + 2 = 10000 + 300 + 2\\ &= 10302\end{aligned}$$

(*iii*) $$\begin{aligned}(999)^2 = (1000-1)^2 &= (1000)^2 - 2 \times (1000) \times 1 + 1^2\\ &= 1000000 - 2000 + 1\\ &= 998001\end{aligned}$$

26. Factorise the following:

(*i*) $4x^2 + 20x + 25$ (*ii*) $9y^2 - 66yz + 121z^2$

(*iii*) $\left(2x + \frac{1}{3}\right)^2 - \left(x - \frac{1}{2}\right)^2$

Sol. (*i*) We have,

$$\begin{aligned}4x^2 + 20x + 25 &= (2x)^2 + 2(2x)(5) + (5)^2\\ &= (2x+5)^2 \qquad [\because\ a^2 + 2ab + b^2 = (a+b)^2]\\ &= (2x+5)(2x+5)\end{aligned}$$

(*ii*) We have,

$$\begin{aligned}9y^2 - 66yz + 121z^2 &= (-3y)^2 + 2(-3y)(11z) + (11z)^2\\ &= (-3y + 11z)^2 \qquad [\because\ a^2 + 2ab + b^2 = (a+b)^2]\\ &= (-3y+11z)(-3y+11z)\\ &= (3y-11z)(3y-11z)\end{aligned}$$

(*iii*) $\left(2x + \frac{1}{3}\right)^2 - \left(x - \frac{1}{2}\right)^2$

Using identity $a^2 - b^2 = (a+b)(a-b)$

$$= \left[\left(2x + \frac{1}{3}\right) + \left(x - \frac{1}{2}\right)\right]\left[\left(2x + \frac{1}{3}\right) - \left(x - \frac{1}{2}\right)\right]$$

$$= \left(2x + \frac{1}{3} + x - \frac{1}{2}\right)\left(2x + \frac{1}{3} - x + \frac{1}{2}\right) = \left(3x - \frac{1}{6}\right)\left(x + \frac{5}{6}\right)$$

27. Factorise the following:

(*i*) $9x^2 - 12x + 3$ (*ii*) $9x^2 - 12x + 4$

Sol. (*i*)
$$\begin{aligned} 9x^2 - 12x + 3 &= 9x^2 - 9x - 3x + 3 \\ &= 9x(x-1) - 3(x-1) \\ &= (9x-3)(x-1) \\ &= 3(3x-1)(x-1) \end{aligned}$$

(*ii*) We have,
$$\begin{aligned} 9x^2 - 12x + 4 &= (3x)^2 - 2(3x)(2) + (2)^2 \\ &= (3x-2)^2 \qquad [\because\ a^2 - 2ab + b^2 = (a-b)^2] \\ &= (3x-2)(3x-2) \end{aligned}$$

28. Expand the following:

(*i*) $(4a - b + 2c)^2$ (*ii*) $(3a - 5b - c)^2$

(*iii*) $(-x + 2y - 3z)^2$

Sol. (*i*) We have,
$$\begin{aligned} (4a - b + 2c)^2 &= (4a)^2 + (-b)^2 + (2c)^2 + 2(4a)(-b) + 2(-b)(2c) + 2(2c)(4a) \\ &\qquad [\because\ a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = (a+b+c)^2] \\ &= 16a^2 + b^2 + 4c^2 - 8ab - 4bc + 16ca \end{aligned}$$

(*ii*) We have,
$$\begin{aligned} (3a - 5b - c)^2 &= (3a)^2 + (-5b)^2 + (-c)^2 + 2(3a)(-5b) \\ &\quad + 2(-5b)(-c) + 2(-c)(3a) \\ &\qquad [\because\ a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = (a+b+c)^2] \\ &= 9a^2 + 25b^2 + c^2 - 30ab + 10bc - 6ca. \end{aligned}$$

(*iii*)
$$\begin{aligned} (-x + 2y - 3z)^2 &= \{(-x) + 2y + (-3z)\}^2 \\ &= (-x)^2 + (2y)^2 + (-3z)^2 + 2(-x)(2y) + 2(2y)(-3z) \\ &\quad + 2(-3z)(-x) \\ &= x^2 + 4y^2 + 9z^2 - 4xy - 12yz + 6zx \end{aligned}$$

29. Factorise the following:

(*i*) $9x^2 + 4y^2 + 16z^2 + 12xy - 16yz - 24xz$

(*ii*) $25x^2 + 16y^2 + 4z^2 - 40xy + 16yz - 20xz$

(*iii*) $16x^2 + 4y^2 + 9z^2 - 16xy - 12yz + 24xz$

Sol. (*i*) We have,
$$\begin{aligned} &9x^2 + 4y^2 + 16z^2 + 12xy - 16yz - 24xz \\ &= (3x)^2 + (2y)^2 + (-4z)^2 + 2(3x)(2y) + 2(2y)(-4z) + 2(-4z)(3x) \\ &= \{3x + 2y + (-4z)\}^2 \quad [\because\ a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = (a+b+c)^2] \\ &= (3x + 2y - 4z)^2 = (3x + 2y - 4z)(3x + 2y - 4z) \end{aligned}$$

(ii) $25x^2 + 16y^2 + 4z^2 - 40xy + 16yz - 20xz$

$$= (-5x)^2 + (4y)^2 + (2z)^2 + 2.(-5x)(4y) + 2(4y)(2z) + 2(2z)(-5x)$$

$$= (-5x + 4y + 2z)^2$$

(iii) We have,

$16x^2 + 4y^2 + 9z^2 - 16xy - 12yz + 24xz$

$= (4x)^2 + (-2y)^2 + (3z)^2 + 2(4x)(-2y) + 2(-2y)(3z) + 2(3z)(4x)$

$= \{4x + (-2y) + 3z\}^2$ $\quad [\because\ a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = (a + b + c)^2]$

$= (4x - 2y + 3z)^2$

$= (4x - 2y + 3z)\ (4x - 2y + 3z)$

30. If $a + b + c = 9$ and $ab + bc + ca = 26$, find $a^2 + b^2 + c^2$.

Sol. We know that

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

$\Rightarrow \quad (a + b + c)^2 = (a^2 + b^2 + c^2) + 2(ab + bc + ca)$

$\Rightarrow \quad 9^2 = (a^2 + b^2 + c^2) + 2(26)$

[Putting the values of $a + b + c$ and $ab + bc + ca$]

$\Rightarrow \quad 81 = (a^2 + b^2 + c^2) + 52$

$\Rightarrow \quad a^2 + b^2 + c^2 = 81 - 52 = 29$

31. Expand the following:

(i) $(3a - 2b)^3$ (ii) $\left(\frac{1}{x} + \frac{y}{3}\right)^3$ (iii) $\left(4 - \frac{1}{3x}\right)^3$

Sol. (i) We have

$$(3a - 2b)^3 = (3a)^3 - (2b)^3 - 3(3a)(2b)(3a - 2b)$$

$[\because\ (a - b)^3 = a^3 - b^3 - 3ab(a - b)]$

$$= 27a^3 - 8b^3 - 18ab\ (3a - 2b)$$

$$= 27a^3 - 8b^3 - 54a^2b + 36ab^2$$

(ii) $\because$ $\quad (x + y)^3 = x^3 + y^3 + 3xy(x + y)$

$\therefore$ $$\left(\frac{1}{x} + \frac{y}{3}\right)^3 = \left(\frac{1}{x}\right)^3 + \left(\frac{y}{3}\right)^3 + 3 \times \frac{1}{x} \times \frac{y}{3}\left(\frac{1}{x} + \frac{y}{3}\right)$$

$$= \frac{1}{x^3} + \frac{y^3}{27} + \frac{y}{x}\left(\frac{1}{x} + \frac{y}{3}\right)$$

$$= \frac{1}{x^3} + \frac{y^3}{27} + \frac{y}{x^2} + \frac{y^2}{3x} = \frac{1}{x^3} + \frac{y}{x^2} + \frac{y^2}{3x} + \frac{y^3}{27}$$

(*iii*) We have,

$$\left(4-\frac{1}{3x}\right)^3 = (4)^3 - \left(\frac{1}{3x}\right)^3 - 3(4)\left(\frac{1}{3x}\right)\left(4-\frac{1}{3x}\right)$$

$$[\because\ (a-b)^3 = a^3 - b^3 - 3ab(a-b)]$$

$$= 64 - \frac{1}{27x^3} - \frac{4}{x}\left(4-\frac{1}{3x}\right)$$

$$= 64 - \frac{1}{27x^3} - \frac{16}{x} + \frac{4}{3x^2}$$

32. Factorise the following:

(*i*) $1 - 64a^3 - 12a + 48a^2$ (*ii*) $8p^3 + \frac{12}{5}p^2 + \frac{6}{25}p + \frac{1}{125}$

Sol. (*i*) We have,

$$1 - 64a^3 - 12a + 48a^2 = (1)^3 - (4a)^3 - 3(1)(4a)(1-4a)$$

$$= (1-4a)^3 \quad [\because\ a^3 - b^3 - 3ab(a-b) = (a-b)^3]$$

$$= (1-4a)(1-4a)(1-4a)$$

(*ii*) $8p^3 + \frac{12}{5}p^2 + \frac{6}{25}p + \frac{1}{125}$

$$= (2p)^3 + 3 \times (2p)^2 \times \frac{1}{5} + 3 \times (2p) \times \left(\frac{1}{5}\right)^2 + \left(\frac{1}{5}\right)^3$$

$$= (2p)^3 + \left(\frac{1}{5}\right)^3 + 3 \times (2p) \times \frac{1}{5}\left[2p + \frac{1}{5}\right]$$

Now, using $a^3 + b^3 + 3ab(a+b) = (a+b)^3$

$$= \left(2p + \frac{1}{5}\right)^3 = \left(2p+\frac{1}{5}\right)\left(2p+\frac{1}{5}\right)\left(2p+\frac{1}{5}\right)$$

33. Find the following products:

(*i*) $\left(\frac{x}{2}+2y\right)\left(\frac{x^2}{4} - xy + 4y^2\right)$ (*ii*) $(x^2-1)(x^4+x^2+1)$

Sol. (*i*) We have,

$$\left(\frac{x}{2}+2y\right)\left(\frac{x^2}{4} - xy + 4y^2\right) = \left(\frac{x}{2}+2y\right)\left\{\left(\frac{x}{2}\right)^2 - \left(\frac{x}{2}\right)(2y) + (2y)^2\right\}$$

$$= \left(\frac{x}{2}\right)^3 + (2y)^3 \qquad [\because\ (a+b)(a^2 - ab + b^2) = a^3 + b^3]$$

$$= \frac{x^3}{8} + 8y^3$$

(*ii*) We have,

$$\begin{aligned}(x^2-1)(x^4+x^2+1) &= (x^2-1)\{(x^2)^2+(x^2)(1)+(1)^2\}\\ &= (x^2)^3-(1)^3\\ &\qquad [\because\ (a-b)(a^2+ab+b^2)=a^3-b^3]\\ &= x^6-1\end{aligned}$$

34. Factorise:

(*i*) $1+64x^3$ (*ii*) $a^3-2\sqrt{2}b^3$

Sol. (*i*) We have,

$$\begin{aligned}1+64x^3 &= (1)^3+(4x)^3\\ &= (1+4x)\ \{(1)^2-(1)(4x)+(4x)^2\}\\ &\qquad [\because\ a^3+b^3=(a+b)(a^2-ab+b^2)]\\ &= (1+4x)(1-4x+16x^2)\\ &= (1+4x)\ (16x^2-4x+1)\\ &= (4x+1)(16x^2-4x+1)\end{aligned}$$

(*ii*) We have,

$$\begin{aligned}a^3-2\sqrt{2}b^3 &= (a)^3-(\sqrt{2}b)^3\\ &= (a-\sqrt{2}b)\{(a)^2+(a)(\sqrt{2}b)+(\sqrt{2}b)^2\}\\ &\qquad [\because\ a^3-b^3=(a-b)(a^2+ab+b^2)]\\ &= (a-\sqrt{2}b)(a^2+\sqrt{2}ab+2b^2)\end{aligned}$$

35. Find the following product:

$(2x-y+3z)\ (4x^2+y^2+9z^2+2xy+3yz-6xz)$

Sol. We have,

$(2x-y+3z)\ (4x^2+y^2+9z^2+2xy+3yz-6xz)$

$=\{2x+(-y)+3z\}\ \{(2x)^2+(-y)^2+(3z)^2-(2x)(-y)-(-y)(3z)-(3z)(2x)\}$

$=(2x)^3+(-y)^3+(3z)^3-3(2x)(-y)(3z)$

$[\because\ (a+b+c)(a^2+b^2+c^2-ab-bc-ca)=a^3+b^3+c^3-3abc]$

$=8x^3-y^3+27z^3+18xyz$

36. Factorise:

(*i*) $a^3-8b^3-64c^3-24abc$ (*ii*) $2\sqrt{2}\,a^3+8b^3-27c^3+18\sqrt{2}abc$

Sol. (*i*) We have,

$$\begin{aligned}&a^3-8b^3-64c^3-24abc\\ &= \{(a)^3+(-2b)^3+(-4c)^3-3(a)(-2b)(-4c)\}\\ &= \{a+(-2b)+(-4c)\}\ \{a^2+(-2b)^2+(-4c)^2-a(-2b)\\ &\qquad -(-2b)(-4c)-(-4c)a\}\end{aligned}$$

$$[\because\ a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)]$$
$$=(a-2b-4c)(a^2+4b^2+16c^2+2ab-8bc+4ca)$$

(*ii*) We have,

$$2\sqrt{2}a^3+8b^3-27c^3+18\sqrt{2}\,abc$$
$$=\{(\sqrt{2}a)^3+(2b)^3+(-3c)^3-3(\sqrt{2}a)(2b)(-3c)\}$$
$$=\{\sqrt{2}a+2b+(-3c)\}\{(\sqrt{2}a)^2+(2b)^2+(-3c)^2-(\sqrt{2}a)(2b)-(2b)(-3c)-(-3c)(\sqrt{2}a)\}$$
$$=(\sqrt{2}a+2b-3c)(2a^2+4b^2+9c^2-2\sqrt{2}ab+6bc+3\sqrt{2}ca)$$

37. Without actually calculating the cubes, find the value of:

(*i*) $\left(\frac{1}{2}\right)^3+\left(\frac{1}{3}\right)^3-\left(\frac{5}{6}\right)^3$ (*ii*) $(0.2)^3-(0.3)^3+(0.1)^3$

Sol. (*i*) Let $a=\frac{1}{2}$, $b=\frac{1}{3}$, $c=-\frac{5}{6}$

$$\therefore\quad a+b+c=\frac{1}{2}+\frac{1}{3}-\frac{5}{6}$$
$$=\frac{3+2-5}{6}=\frac{0}{6}=0$$
$$\Rightarrow\quad a^3+b^3+c^3=3abc$$
$$\therefore\quad \left(\frac{1}{2}\right)^3+\left(\frac{1}{3}\right)^3-\left(\frac{5}{6}\right)^3=\left(\frac{1}{2}\right)^3+\left(\frac{1}{3}\right)^3+\left(-\frac{5}{6}\right)^3$$
$$=3\times\frac{1}{2}\times\frac{1}{3}\left(-\frac{5}{6}\right)=-\frac{5}{12}$$

(*ii*) We have,

$(0.2)^3-(0.3)^3+(0.1)^3=(0.2)^3+(-0.3)^3+(0.1)^3$

Let $a=0.2$, $b=-0.3$ and $c=0.1$. Then,

$$a+b+c=0.2+(-0.3)+0.1$$
$$=0.2-0.3+0.1=0$$

$\because\quad a+b+c=0$

$\therefore\ a^3+b^3+c^3=3abc$

$\Rightarrow\quad (0.2)^3+(-0.3)^3+(0.1)^3=3(0.2)(-0.3)(0.1)=-0.018$

Hence, $(0.2)^3+(-0.3)^3+(0.1)^3=-0.018$

38. Without finding the cubes, factorise
$(x-2y)^3+(2y-3z)^3+(3z-x)^3$

Sol. Let $x-2y=a$, $2y-3z=b$ and $3z-x=c$

$$\therefore \quad a+b+c = x-2y+2y-3z+3z-x=0$$

$$\Rightarrow \quad a^3+b^3+c^3 = 3abc$$

Hence, $(x-2y)^3+(2y-3z)^3+(3z-x)^3$

$$= 3(x-2y)(2y-3z)(3z-x)$$

39. Find the value of

(*i*) $x^3+y^3-12xy+64$, when $x+y=-4$

(*ii*) $x^3+8y^3-36xy-216$, when $x=2y+6$

Sol. (*i*) $x^3+y^3-12xy+64 = x^3+y^3+4^3-3x\times y\times 4$

$$= (x+y+4)(x^2+y^2+4^2-xy-4y-4x)$$

$$[\because x+y=-4]$$

$$= (0)(x^2+y^2+4^2-xy-4y-4x)=0$$

(*ii*) $x^3-8y^3-36xy-216 = x^3+(-2y)^3+(-6)^3-3x(-2y)(-6)$

$$= (x-2y-6)$$
$$[x^2+(-2y)^2+(-6)^2-x(-2y)-(-2y)(-6)-(-6)x]$$
$$= (x-2y-6)(x^2+4y^2+36+2xy-12y+6x)$$
$$= (0)(x^2+4y^2+36+2xy-12y+6x)=0$$

$$[\because x=2y+6]$$

40. Give possible experiments for the length and breadth of the rectangle whose area is given by $4a^2+4a-3$.

Sol. Area : $4a^2+4a-3$

Using the method of splitting the middle term, we first find two numbers whose sum is +4 and product is $4\times(-3)=-12$.

Now, $+6-2=+4$ and $(+6)\times(-2)=-12$

We split the middle term $4a$ as $4a=+6a-2a$,

so that $\quad 4a^2+4a-3 = 4a^2+6a-2a-3$

$$= 2a(2a+3)-1(2a+3)$$
$$= (2a-1)(2a+3)$$

Now, area of rectangle $=4a^2+4a-3$

Also, area of rectangle = length × breadth and $4a^2+4a-3=(2a-1)(2a+3)$

So, the possible expressions for the length and breadth of the rectangle are length $=(2a-1)$ and breadth $=(2a+3)$ or, length $=(2a+3)$ and breadth $=(2a-1)$.

EXERCISE 2.4

1. If the polynomials $az^3 + 4z^2 + 3z - 4$ and $z^3 - 4z + a$ leave the same remainder when divided by $z - 3$, find the value of a.

Sol. Let $p(z) = az^3 + 4z^2 + 3z - 4$

and $\quad q(z) = z^3 - 4z + a$

As these two polynomials leave the same remainder, when divided by $z - 3$, then $p(3) = q(3)$.

$\therefore \quad p(3) = a(3)^3 + 4(3)^2 + 3(3) - 4$
$\qquad = 27a + 36 + 9 - 4$

or $\quad p(3) = 27a + 41$

and $\quad q(3) = (3)^3 - 4(3) + a$
$\qquad = 27 - 12 + a = 15 + a$

Now, $\quad p(3) = q(3)$

$\Rightarrow \quad 27a + 41 = 15 + a$

$\Rightarrow \quad 26a = -26;\ a = -1$

Hence, the required value of $a = -1$.

2. The polynomial $p(x) = x^4 - 2x^3 + 3x^2 - ax + 3a - 7$ when divided by $x + 1$ leaves remainder 19. Also, find the remainder when $p(x)$ is divided by $x + 2$.

Sol. We know that if $p(x)$ is divided by $x + a$, then the remainder $= p(-a)$.

Now, $p(x) = x^4 - 2x^3 + 3x^2 - ax + 3a - 7$ is divided by $x + 1$, then the remainder $= p(-1)$

Now, $\quad p(-1) = (-1)^4 - 2(-1)^3 + 3(-1)^2 - a(-1) + 3a - 7$
$\qquad = 1 - 2(-1) + 3(1) + a + 3a - 7$
$\qquad = 1 + 2 + 3 + 4a - 7$
$\qquad = -1 + 4a$

Also, remainder $= 19$

$\therefore \quad -1 + 4a = 19$

$\Rightarrow \quad 4a = 20;\ a = 20 \div 4 = 5$

Again, when $p(x)$ is divided by $x + 2$, then

remainder $= p(-2) = (-2)^4 - 2\,(-2)^3 + 3(-2)^2 - a(-2) + 3a - 7$
$\qquad = 16 + 16 + 12 + 2a + 3a - 7$
$\qquad = 37 + 5a$
$\qquad = 37 + 5(5) = 37 + 25 = 62$

3. If both $(x - 2)$ and $\left(x - \frac{1}{2}\right)$ are factors of $px^2 + 5x + r$, show that $p = r$.

Sol. Let $p(x) = px^2 + 5x + r$.

As $(x-2)$ is a factor of $p(x)$

So, $p(2) = 0 \Rightarrow p(2)^2 + 5(2) + r = 0$

$\Rightarrow \quad 4p + 10 + r = 0$ ...(1)

Again, $\left(x - \frac{1}{2}\right)$ is a factor of $p(x)$,

$\therefore \quad p\left(\frac{1}{2}\right) = 0$

Now, $p\left(\frac{1}{2}\right) = p\left(\frac{1}{2}\right)^2 + 5\left(\frac{1}{2}\right) + r$

$= \frac{1}{4}p + \frac{5}{2} + r$

$\therefore \quad p\left(\frac{1}{2}\right) = 0 \Rightarrow \frac{1}{4}p + \frac{5}{2} + r = 0$ ...(2)

From (1), we have $4p + r = -10$

From (2), we have $p + 10 + 4r = 0$

$\Rightarrow \quad p + 4r = -10$

$\therefore \quad 4p + r = p + 4r$ [$\because$ Each $= -10$]

$\therefore \quad 3p = 3r \Rightarrow p = r$

Hence, proved.

4. Without actual division, prove that $2x^4 - 5x^3 + 2x^2 - x + 2$ is divisible by $x^2 - 3x + 2$.

Sol. We have,

$$\begin{aligned} x^2 - 3x + 2 &= x^2 - x - 2x + 2 \\ &= x(x-1) - 2(x-1) \\ &= (x-1)(x-2) \end{aligned}$$

Let $p(x) = 2x^4 - 5x^3 + 2x^2 - x + 2$

Now, $p(1) = 2(1)^4 - 5(1)^3 + 2(1)^2 - 1 + 2 = 2 - 5 + 2 - 1 + 2 = 0$

Therefore, $(x-1)$ divides $p(x)$

and $p(2) = 2(2)^4 - 5(2)^3 + 2(2)^2 - 2 + 2$

$= 32 - 40 + 8 - 2 + 2 = 0$

Therefore, $(x-2)$ divides $p(x)$.

So, $(x-1)(x-2) = x^2 - 3x + 2$ divides $2x^4 - 5x^3 + 2x^2 - x + 2$

5. Simplify $(2x - 5y)^3 - (2x + 5y)^3$.

Sol. We have,

$(2x - 5y)^3 - (2x + 5y)^3$

$= \{(2x - 5y) - (2x + 5y)\}\{(2x - 5y)^2 + (2x - 5y)(2x + 5y) + (2x + 5y)^2\}$

[$\because \; a^3 - b^3 = (a-b)(a^2 + ab + b^2)$]

$= (2x - 5y - 2x - 5y)(4x^2 + 25y^2 - 20xy + 4x^2 - 25y^2 + 4x^2 + 25y^2 + 20xy)$

$= (-10y)(2x^2 + 25y^2)$

$= -120x^2y - 250y^3$

6. Multiply $x^2 + 4y^2 + z^2 + 2xy + xz - 2yz$ by $(-z + x - 2y)$.

Sol. We have,

$(-z + x - 2y)(x^2 + 4y^2 + z^2 + 2xy + xz - 2yz)$

$= \{x + (-2y) + (-z)\}\{(x)^2 + (-2y)^2 + (-z)^2 - (x)(-2y) - (-2y)(-z) - (-z)(x)\}$

$= x^3 + (-2y)^3 + (-z)^3 - 3(x)(-2y)(-z)$

$[\because (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = a^3 + b^3 + c^3 - 3abc]$

$= x^3 - 8y^3 - z^3 - 6xyz$

7. If a, b, c are all non-zero and $a + b + c = 0$, prove that

$$\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} = 3.$$

Sol. We have a, b, c are all non-zero and $a + b + c = 0$, therefore

$$a^3 + b^3 + c^3 = 3abc$$

Now, $$\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} = \frac{a^3 + b^3 + c^3}{abc} = \frac{3abc}{abc} = 3$$

8. If $a + b + c = 5$ and $ab + bc + ca = 10$, then prove that $a^3 + b^3 + c^3 - 3abc = -25$

Sol. We know that,

$$\begin{aligned} a^3 + b^3 + c^3 - 3abc &= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \\ &= (a + b + c)[a^2 + b^2 + c^2 - (ab + bc + ca)] \\ &= 5\{a^2 + b^2 + c^2 - (ab + bc + ca)\} \\ &= 5(a^2 + b^2 + c^2 - 10) \end{aligned}$$

Now, $a + b + c = 5$

Squaring both sides, we get

$$(a + b + c)^2 = 5^2$$

$\Rightarrow a^2 + b^2 + c^2 + 2(ab + bc + ca) = 25$

$\therefore \quad a^2 + b^2 + c^2 + 2(10) = 25$

$\Rightarrow \quad a^2 + b^2 + c^2 = 25 - 20 = 5$

Now, $$\begin{aligned} a^3 + b^3 + c^3 - 3abc &= 5(a^2 + b^2 + c^2 - 10) \\ &= 5(5 - 10) = 5(-5) = -25 \end{aligned}$$

Hence, proved.

9. Prove that $(a + b + c)^3 - a^3 - b^3 - c^3 = 3\,(a + b)\,(b + c)\,(c + a)$

Sol.

$$
\begin{aligned}
(a+b+c)^3 &= [a+(b+c)]^3 \\
&= a^3 + 3a^2(b+c) + 3a(b+c)^2 + (b+c)^3 \\
&= a^3 + 3a^2b + 3a^2c + 3a(b^2 + 2bc + c^2) \\
&\quad + (b^3 + 3b^2c + 3bc^2 + c^3) \\
&= a^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc + 3ac^2 \\
&\quad + b^3 + 3b^2c + 3bc^2 + c^3 \\
&= a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3b^2c + 3b^2a \\
&\quad + 3c^2a + 3c^2b + 6abc \\
&= a^3 + b^3 + c^3 + 3a^2\,(b+c) + 3b^2(c+a) \\
&\qquad\qquad + 3c^2\,(a+b) + 6abc
\end{aligned}
$$

Hence, above result can be put in the form

$$(a+b+c)^3 = a^3 + b^3 + c^3 + 3\,(a+b)\,(b+c)\,(c+a)$$

$\therefore\ (a + b + c)^3 - a^3 - b^3 - c^3 = 3(a + b)\,(b + c)\,(c + a)$