RELATION AND FUNCTIONS

RELATION

Let A and B be two non-empty sets, then every subset of A × B defines a relation from A to B and every relation from A to B is a subset of A × B.

Let R $\subseteq$ A × Band (a, b) $\in$R. Then we say that a is related to b by the relation R and write it as aRb. If (a, b) $\in$ R, we write it as aRb.

Example: Let A = {1, 2, 5, 8, 9}, B = {1, 3} we set a relation from A to B as: a R b iff a $\leq$ b; a $\in$ A, b $\in$ B. Then R = {(1, 1)}, (1, 3), (2, 3)} $\subset$ A × B

(a) **Total number of relations:** Let A and B be two non-empty finite sets consisting of m and n elements respectively. Then A × B consists of mn ordered pairs. So, total number of subset of A × B is 2^{mn}. Since each subset of A × B defines relation from A to B, so total number of relations from A to B is 2^{mn}.

(b) **Domain of a relation:** Let R be a relation from a set A to a set B. Then the set of all first components or coordinates of the ordered pairs belonging to R is called the domain of R.

Thus, Domain (R) = {a : (a, b) $\in$ R}

(c) **Range of a relation:** Let R be a relation from a set A to a set B. Then the set of all second components or coordinates of the ordered pairs in R is called the range of R.

Thus, Range (R) = {b : (a, b) $\in$ R}.

The domain of a relation from A to B is a subset of A and its range is a subset of B.

(d) **Codomain of a relation:** If ‘R’ be a relation defined from set A to set B, then ‘B’ is called co-domain of relation ‘R’.

(e) **Relation on a set:** Let A be a non-void set. Then, a relation from A to itself i.e. a subset of A × A is called a relation on set A.

SOLVED EXAMPLE

1. Let *A* = {1, 2, 3}. The total number of distinct relations that can be defined over *A* is

(a) 2^9 (b) 6 (c) 8 (d) None of these

Sol. (a) $n(A \times A) = n(A).n(A) = 3^2 = 9$

So, the total number of subsets of $A \times A$ is a relation over the set *A*.

$\Rightarrow$ Total number of distinct relations = 2^9.

2. A = {2, 4, 6, 9} and B = {4, 6, 18, 27, 54}. Find a relation R from A to B, such that for any a $\in$ A and b $\in$ B, 'a' is factor of 'b' and a < b.

Sol. Since A = {2, 4, 6, 9} and B = {4, 6, 18, 27, 54},

we have to find the set of ordered pairs (a, b) such that a is factor of b and a < b.

Let 2 $\in$ A, 4 $\in$ B, as 2 is a factor of 4 and 2 < 4. So (2, 4) is one such ordered pair.

Likewise, (2, 6), (2, 18), (2, 54)… are other such ordered pairs.

Thus, the required relation is

R = {(2, 4), (2, 6), (2, 18), (2, 54), (6, 18), (6, 54,), (9, 18), (9, 27), (9, 54)}.

Domain of R = {2, 6, 9} ; Range of R = {4, 6, 18, 27, 54}

PRACTICE WORKSHEET

1. A relation R from A to B is an arbitrary subset of

a. A × B   b. B × A   c. A × A   b. B × B

2. If (a, b) ∈ R, where R is a relation, then

a. (a, b) is not an ordered pair   b. a = b only

c. ab = 1 only   d. a is related to b under the relation R

3. A relation on a set A is a subset of

a. A × A   b. A   c. A . A   d. None of these

ANSWERS

1. a   2. d   3. a

PREVIOUS YEARS BOARD QUESTIONS

1. If R = {(a, a^3) : a is a prime number less than 5} be a relation. Find the range of R. **[Foreign 2014]**

2. If R = {(x, y) : x + 2y = 8} is a relation on N, then write the range of R. **[AI 2014]**

ANSWERS

1. {8, 27}   2. {1, 2, 3}

1.1 TYPES OF RELATIONS DEFINED FROM ONE SET TO ANOTHER SET

A. EMPTY RELATION

A relation R from set A to set B is called empty relation, if there exist no (a, b) ∈ R where a ∈ A and b ∈ B.

Example : Let A = {2, 4} and B = {1, 3, 5} then R = {(a, b) : a ∈ A, b ∈ B and a –b is even} is empty or void relation.

B. UNIVERSAL RELATION

A relation R in a set A is called a universal relation or total relation, if every element of A is related to every element of A, i.e. R = A × A.

An empty relation and the universal relation are called *trivial relations*.

C. INVERSE RELATION

Let A, B be two sets and let R be a relation from a set A to a set B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by R^{-1} = {(b, a) : (a, b) ∈ R}.

Clearly (a, b) ∈ R ⇔ (b, a) ∈ R^{-1} .

Also, Domain (R) = Range (R^{-1}) and Range (R) = Domain (R^{-1})

Example: Let A = {a, b, c}, B = {1, 2, 3} and R = {(a, 1), (a, 3), (b, 3), (c, 3)}.

Then,

(i) R^{-1} = {(1, a), (3, a), (3, b), (3, c)}

(ii) Domain (R) = {a, b, c} = Range (R^{-1})

(iii) Range (R) = {1, 3} = Domain (R^{-1})

Note : $(R^{-1})^{-1}$ = R.

1.2 TYPES OF RELATIONS DEFINED ON A SET

A. IDENTITY RELATION: Let A be a set.

Then the relation I_A = {(a, a) : a $\in$ A} on A is called the identity relation on A.

In other words, a relation I_A on A is called the identity relation if every element of A is related to itself only.

Example: On the set A = {1, 2, 3}, R = {(1, 1), (2, 2), (3, 3)} is the identity relation on A .

R = {(1, 1), (2, 2), } is not Identity relation on set A because 3 $\in$ A but (3, 3) $\notin$ A.

B. REFLEXIVE RELATION: A relation R on a set A is said to be reflexive if every element of A is related to itself.

Thus, R is reflexive $\Leftrightarrow$ (a, a) $\in$ R for all a $\in$ A.

A relation R on a set A is not reflexive if there exists an element a $\in$ A such that (a, a) $\notin$ R.

Note: It is interesting to note that every identity relation is reflexive but every reflexive relation need not be an identity relation.

Example: Let A = {1, 2, 3} and R = {(1, 1), (2, 2)}

Then R is not reflexive since but (3, 3) $\notin$ R.

However R={(1,1), (2,2), (3,3) } and R = {(1, 1), (2, 2), (3, 3), (1,3) } are reflexive relation on set A.

C. SYMMETRIC RELATION: A relation R on a set A is said to be a symmetric relation iff

(a, b) $\in$ R $\Rightarrow$ (b, a) $\in$ R for all a, b $\in$ A i.e. aRb $\Rightarrow$ bRa for all a, b $\in$ A.

it should be noted that R is symmetric iff $R^{-1} = R$

D. TRANSITIVE RELATION: Let A be any set. A relation R on set A is said to be a transitive relation iff

(a, b) $\in$ R and (b, c) $\in$ R $\Rightarrow$ (a, c) $\in$ R for all a, b, c $\in$ A i.e., aRb & bRc $\Rightarrow$ aRc for all a, b, c $\in$ A.

In other words, if a is related to b, b is related to c, then a is related to c.

Transitivity fails only when there exists a, b, c such that (a, b) $\in$ R and (b, c) $\in$ R but (a, c) $\notin$ R.

Example: Consider the set A = {1, 2, 3} and the relations

$R_1 = \{(1, 2), (1, 3)\}$; R_2= {(1, 2), (2, 3), (1, 3)}; R_3 = {(1, 1)}; R_4 = {(1, 2), (2, 1), (1, 1)}

Then R_1, R_2, R_3 are transitive while $R_4$is not transitive since $R_4, (2, 1) \in R_4; (1, 2) \in R_4$ in but $(2, 2) \notin R_4$.

The relation 'is congruent to' on the set T of all triangles in a plane is a transitive relation.

E. EQUIVALENCE RELATION: A relation R on a set A is said to be an equivalence relation on A iff

(i) It is reflexive i.e. (a, a) $\in$ R for all a $\in$ A

(ii) It is symmetric i.e. (a, b) $\in$ R $\Rightarrow$ (b, a) $\in$ R, for all a, b $\in$ A

(iii) It is transitive i.e. (a, b) $\in$ R and (b, c) $\in$ R $\Rightarrow$ (a, c) $\in$ R for all a, b, c $\in$ A.

Note: Every identity relation will be reflexive, symmetric and transitive.

IMPORTANT TIPS

- ☞ If R and S are two equivalence relations on a set A , then R $\cap$ S is also an equivalence relation on A.
- ☞ The union of two equivalence relations on a set is not necessarily an equivalence relation on the set.
- ☞ If R is an equivalence relation on a set A, then $R^{-1}$is also an equivalence relation on A.

F. EQUIVALENCE CLASSES OF AN EQUIVALENCE RELATION: Let R be equivalence relation in $A(\neq \varphi)$. Let $a \in A$ then the equivalence class of a, denoted by [a] or is defined as the set of all those points of A which are related to a under the relation R.

Thus [a] = {x $\in$ A : x R a}.

1. A relation R in a set A is called empty relation, if
a. no element of A is related to any element of A
b. every element of A is related to every element of A
c. some elements of A are related to some elements of A
d. None of the above

2. A relation R in a set A is called universal relation, if
a. each element of A is not related to every element of A
b. no element of A is related tyo any element of A
c. each element of A is related to every element of A
d. None of the above

3. The trivial relation(s) is/are
a. empty relation only
b. universal relation only
c. empty relation and universal relation
d. None of the above

4. If R is a relation in a set A such that $(a, a) \in R$ for every $a \in A$, then the relation R is called
a. symmetric b. reflexive c. transitive d. symmetric or transitive

5. A relation R in a set A is called symmetric, if for all $a_1, a_2 \in A$.
a. $(a_1, a_2) \in R \Rightarrow (a_2, a_1) \in R$
b. $(a_1, a_2) \in R \Rightarrow (a_1, a_1) \in R$
c. $(a_1, a_2) \in R \Rightarrow (a_2, a_2) \in R$
d. None of these

6. A relation R in a set A is called transitive, if for all $a_1, a_2, a_3 \in A$, $(a_1, a_2) \in R$ and $(a_2, a_3) \in R$ implies
a. $(a_2, a_1) \in R$ b. $(a_1, a_3) \in R$ c. $(a_3, a_1) \in R$ d. $(a_3, a_2) \in R$

7. If R = {(x, y): x and y work at the same place}, then R is
a. reflexive b. symmetric c. transitive d. an equivalence relation

8. If Rs = {(x, y): a and y live in the same locality}, then R is
a. not reflexive b. not transitive c. not symmetric d. an equivalence relation

9. If R = {(x, y): x is exactly 7cm taller than y}, then R is
a. not symmetric b. reflexive c. symmetric but not transitive d. an equivalence relation

10. If R = {(x, y): x is wife of y}, then R is
a. reflexive b. symmetric c. transitive d. an equivalence relation

11. If R = {(x, y): x father of y}, then R is
a. reflexive but not symmetric
b. symmetric and transitive
c. neither reflexive nor symmetric nor transitive
d. symmetric but not reflexive

12. The relation R defined in the set A = {1, 2, 3, 4, 5, 6, 7} by R = {(a, b): both a and b are either odd or even}. Then, R is
a. symmetric b. transitive c. an equivalence relation d. reflexive

13. Let R be the relation defined in the set A = {1, 2, 3, 4, 5, 6, 7} by R = {(a, b): both a and b are either odd or even}. Now, consider the following statements
I. All the elements of the subset {1, 3, 5, 7} are related to each other.
II. All the elements of the subset {2, 4, 6} are related to each other.
III. Some elements of the subset {1, 3, 5, 7} are related to some elements of the subset {2, 4, 6}.
Then
a. I and II are true b. I and II are true c. II and III are true d. All are true

14. If A = {x $\in$ Z : 0 $\le$ x $\le$ 12} and R is the relation in A given by R = {(a, b) : a = b}. Then, the set of all elements related to 1 is:

a. {2, 3} b. {2, 3} c. {1} d. {2}

15. If R_1 and R_2 are equivalence relation in a set A, then $R_1 \cap R_2$ is

a. symmetric b. reflexive c. transitive d. an equivalence relation

16. Let W denote the words in the English dictionary. Define the relation R as follows :

R = {(x, y) $\in$ W × W : the words x and y have atelast one letter in common}.

Then, R is

a. not reflexive, symmetric and transitive b. reflexive, symmetric and not transitive

c. reflexive, not symmetric and transitive d. reflexive, symmetric and transitive

17. Let A = {1, 2, 3}. Then, the number of equivalence relations containing (1, 2) is

a. 1 b. 2 c. 3 d. 4

18. Given a non-empty set X, consider P(X) which is the set of all subsets of X.

Define the relation R in P(X) as follows :

For subsets A and B in P(X), ARB, if and only if A $\subset$ B. Then, R is

a. reflexive b. transitive c. not symmetric d. all of these

ANSWERS

1. a	**2.** c	**3.** c	**4.** b	**5.** a	**6.** b	**7.** d	**8.** d	**9.** a
10. c	**11.** c	**12.** c	**13.** b	**14.** c	**15.** d	**16.** b	**17.** b	**18.** d

ASSIGNMENT

1. Give examples of a relation such that R is defined on A = {1, 2, 3}, which is

i. Symmetric and transitive but not reflexive **ii.** Transitive but neither reflexive nor symmetric

iii. Reflexive and sym metric but not transitive **iv.** Reflexive and transitive but not symmetric

v. Symmetric but neither reflexive nor transitive

2. Comment on reflexive, symmetric and transitive nature of R, if R is a relation on **A.**

i. A = {a, b, c}, If R = {(a,b), (b,a), (a,c), (c,a)} **ii.** A = {1,2,3} & R = {(1,1), (2,2), (3,3), (1,2), (2,3)}

iii. A = {a, b, c}, If R = {(a, a), (a, b), (a, c), (b, b), (b, c), (c, a), (c, b), (c, c)}

iv. A = {1, 2, 3, 4} given by R = {(1, 2), (2, 2), (1,1), (4, 4), (1, 3), (3, 3), (3, 2)}

3. Given relation R = {(1, 2), (1, 1), (2, 3)} on the set A = {1, 2, 3}, minimum number of order pairs may be added to R so that it becomes a transitive relation on A.

4. Given relation R = {(1, 2), (2, 3)} on the set A = {1, 2, 3}, minimum number of order pairs may be added to R so that it becomes an equivalence relation on A.

5. Let A = {1, 2, 3}. Find number of relations containing (1, 2) and (2, 3) which are reflexive and transitive but not symmetric.

6. Find that the number of equivalence relation on the set {1, 2, 3} containing (1, 2) & (2,1).

7. Let A = {1, 2, 3}. Find number of relations containing (1, 2) and (1, 3) which are reflexive and symmetric but not transitive.

8. Comment on reflexive, symmetric and transitive nature of R, if R is on A.

i. Let A be the set of all straight lines drawn in a plane and R be the relation '*is perpendicular to*' on A,

ii. A = {1,2,3, 4, 5, 6} as R = {(x, y): y = x + 1} **iii.** A = {1,2,3,---, 14} as R = {(x, y): 3x – y = 0}

iv. A = {1, 2, 3, 4, 5, 6} as R = {(x, y): y is divisible by x}

9. Comment on reflexive, symmetric and transitive nature of R if R is on

i. R and **R** = {(a, b): a $\leq$ b} **ii.** R and **R** = {(a, b): a $\leq$ b^2}

iii. R and **R** = {(a, b): a $\leq$ b^3} **iv.** Z and **R** = {(x, y): x – y is an integer}

10. Show that the relation R on the set

i. A=$\{x \in Z : 0 \le x \le 12\}$, given by $R = \{(a,b) : |a-b|\ is\ a\ multiple\ of\ 4\}$ is an equivalence relation . Also find the set of all elements related to 1.

ii. A = $\{x \in Z : 0 \le x \le 12\}$, given by $R = \{(a,b) : a = b\}$ is an equivalence relation

11. Let Z be the set of all integers and R be the relation on Z defined as R = {(a,b) : a, b $\in$ Z, & (a –b) is divisible by 5}. Prove that R is an equivalence relation.

12. Let R be a relation defined in the set A = {1, 2, 3, 4, 5} by R = {(a, b): $|a-b|$ is even}. Show that R is an equivalence relation. Further, show that all elements of subsets {1, 3, 5} are related to each other and all the elements of the subset {2, 4} are related to each other, but no element of subset {1, 3,5} is related to any element of subset {2, 4}.Write the equivalence class {1}.

13. Let R be a relation defined in the set A = {1, 2, 3, 4, 5, 6, 7} by R = {(a, b): both a and b are either even or odd}. Show that R is an equivalence relation. Further, show that all elements of subsets {1, 3, 5, 7} are related to each other and all the elements of the subset {2, 4, 6} are related to each other, but no element of subset {1, 3, 5, 7} is related to any element of subset {2, 4, 6}.

14. Show that the relation R in the set A of points in a plane given by R = {T_1, T_2 : T_1 is similar to T_2}, is equivalence relation. Consider three right angle triangles T_1 with sides 3, 4, 5, T_2 with sides 5, 12, 13 and T_3 with sides 6, 8, 10. Which triangles among T_1, T_2 and T_3 are related?

15. Prove that the relation R on set N$\times$N is an equivalence relation defined by

i. $(a,b)R(c,d) \Leftrightarrow a+d = b+c$ if (a, b), (c, d) $\in$ N × N.

ii. $(a,b)R(c,d) \Leftrightarrow ad(b+c) = bc(a+d)$ if (a, b), (c, d) $\in$ N × N.

16. Let X = {1, 2, 3, 4, 5, 6, 7, 8, 9}. Let R_1 be a relation in X given by R_1 = {(x, y): x– y is $\div$ by 3} and R_2 be another relation on X given by R_2 = {(x, y) : {x, y}$\subset${1, 4, 7} or {x, y}$\subset${2, 5, 8} or {x, y}$\subset${3, 6, 9}}. Show that R_1 = R_2.

17 The relation R from set A = $\{1,2,4\}$ to B = $\{0,3\}$ is given by R = $\{(1,3),(2,0),(4,3)\}$. Find R^{-1}.

18. The relation R on set A = {1, 2, 3, 4, 5, 6} defined as R={(x,y):x is divisible by y}. Find R and hence find R^{-1}.

ANSWERS

1. (A) R = {(1, 3), (3, 1), (1, 1)}, (B) R = {(1, 1), (3, 3), (1, 3), (3, 1)},(C) R = {(1, 1), (2, 2), (3, 3), (1, 3), (3, 1), (2, 3), (3, 2)} (D) R = {(1, 1), (2, 2), (3, 3), (1, 3), (E) R = {(1, 3), (3, 1)}

2. i. Symmetric **ii.** Reflexive **iii.** Reflexive **iv.** Reflexive, Transitive

3. (1, 3) **4.** (1, 1), (2, 2), (3, 3), (1, 3), (2 1), (3, 2), (3, 1)

5. 3 **6.** 2 **7.** 1

8. i. Symmetric **ii.** Not any type **iii.** Not any type **iv.** Reflexive, Transitive

9. i. Reflexive, Transitive **ii.** Not any type **iii.** Not any type **iv.** Equivalence

10. i. {(1, 1), (1, 5), (1, 9)} **12.** {1, 3, 5} equivalence class of {1} **14.** T_1 and T_3

17. R^{-1} = {(3, 1), (0, 2), (3, 4)}

18. R^{-1} = {(1, 1), (1, 2), (2, 2), (1, 3), (3, 3), (1, 4), (2, 4), (4, 4), (1, 5), (5, 5), (1, 6), (2, 6), (3, 6), (6, 6)}

PREVIOUS YEARS BOARD QUESTIONS

1. Let R is the equivalence relation in the set A = {0, 1, 2, 3, 4, 5} given by R = {(a, b) : 2 divides (a –b)}. Write the equivalence class [0]. **[Delhi 2014C]**

2. State the reason for the relation R in the set {1, 2, 3} given by R = {(1, 2), (2, 1)} not to be transitive. **[Delhi 2011]**

3. If R is a relation defined on the set of natural numbers N as follows :

R = {(x, y), x $\in$ N, y $\in$ N and 2x + y = 24}, then find the domain and range of the relation R. Also, find if R is an equivalence relation or not. **[Delhi 2014]**

4. If Z is the set of all integers and R is the relation on Z defined as R = {(a, b) : a, b $\in$ Z and a –b is divisible by 5}. Prove that R is an equivalence relation. **[Delhi 2010]**

5. Show that the relation S is the set R of real numbers defined as, S = {(a, b) : a, b $\in$ R and a $\leq b^3$} is neither reflexive nor symmetric nor transitive. **[Delhi 2010]**

6. Show that the relation S in set A = {x $\in$ Z : 0 $\leq$ x $\leq$ 12} given by S = {(a, b) : a, b $\in$ Z, | a –b| is divisible by 4} is an equivalence relation. Find the set of all elements related to A. **[AI 2010]**

7. Show that the relation S defined on set N × N be (a, b) S (c, d) $\Rightarrow$ a + d = b + c is an equivalence relation. **[AI 2010]**

8. Prove that the relation R in set A = {1, 2, 3, 4, 5} given by R = {(a, b) : | a –b | is even} is an equivalence relation. **[Delhi 2009]**

ANSWERS

1. {0, 2, 4} **3.** Domain of R = {1, 2, 3 .... , 11} Range of R = {2, 4, 6, 8, 10, 12, 14, 16 , 18, 20 , 22}

EXEMPLAR PROBLEMS

SHORT ANSWER

1. Let A = {*a*, *b*, *c*} and the relation R be defined on A as follows: R = {(*a*, *a*), (*b*, *c*), (*a*, *b*)}. Then, write minimum number of ordered pairs to be added in **R** to make **R** reflexive and transitive.

2. Let *n* be a fixed positive integer. Define a relation R in **Z** as follows: *a, b* ***Z***, *a*R*b* if and only if *a* – *b* is divisible by *n*. Show that R is an equivalance relation.

LONG ANSWER

3. If A = {1, 2, 3, 4 }, define relations on A which have properties of being:

(a) reflexive, transitive but not symmetric

(b) symmetric but neither reflexive nor transitive

(c) reflexive, symmetric and transitive.

4. Let R be relation defined on the set of natural number **N** as follows: R = {(*x*, *y*): $x \in$ **N**, $y \in$ **N**, 2*x* + *y* = 41}. Find the domain and range of the relation R. Also verify whether R is reflexive, symmetric and transitive.

5. Each of the following defines a relation on **N**:

(i) *x* is greater than *y*, *x, y* $\in$ **N**

(ii) *x* + *y* = 10, *x, y* $\in$ **N**

(iii) *x y* is square of an integer *x, y* $\in$ **N**

(iv) *x* + 4y = 10 *x, y* $\in$ **N**.

Determine which of the above relations are reflexive, symmetric and transitive.

6. Let A = {1, 2, 3, ... 9} and R be the relation in A × A defined by (*a*, *b*) R (*c*, *d*) if *a* + *d* = *b* + *c* for (*a*, *b*), (*c*, *d*) in A × A. Prove that R is an equivalence relation and also obtain the equivalent class [(2, 5)].

OBJECTIVE TYPE QUESTIONS

Choose the correct answer out of the given four options for questions from 7 to 12 (M.C.Q.).

7. Let T be the set of all triangles in the Euclidean plane, and let a relation R on T be defined as aRb if a is congruent to b $\forall$ $a, b \in$ T. Then R is :

(A) reflexive but not transitive
(B) transitive but not symmetric
(C) equivalence
(D) none of these

8. Consider the non-empty set consisting of children in a family and a relation R defined as aRb if a is brother of b. Then R is

(A) symmetric but not transitive
(B) transitive but not symmetric
(C) neither symmetric nor transitive
(D) both symmetric and transitive

9. The maximum number of equivalence relations on the set A = {1, 2, 3} are

(A) 1
(B) 2
(C) 3
(D) 5

10. If a relation R on the set {1, 2, 3} be defined by R = {(1, 2)}, then R is

(A) reflexive
(B) transitive
(C) symmetric
(D) none of these

11. Let us define a relation R in **R** as aRb if $a \geq b$. Then R is

(A) an equivalence relation
(B) reflexive, transitive but not symmetric
(C) symmetric, transitive but not reflexive
(D) neither transitive nor reflexive but symmetric.

12. Let A = {1, 2, 3} and consider the relation R = {1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1,3)}. Then R is

(A) reflexive but not symmetric
(B) reflexive but not transitive
(C) symmetric and transitive
(D) neither symmetric, nor transitive

FILL IN THE BLANKS

13. Let the relation R be defined in **N** by aRb if $2a + 3b = 30$. Then R = ___________.

14. Let the relation R be defined on the set A={1, 2, 3, 4, 5} by R={(a, b): $|a^2 - b^2| < 8$. Then R is given by _______.

TRUE OR FALSE

15. Let R = {(3, 1), (1, 3), (3, 3)} be a relation defined on the set A = {1, 2, 3}. Then R is symmetric, transitive but not reflexive.

16. Every relation which is symmetric and transitive is also reflexive.

17. An integer m is said to be related to another integer n if m is a integral multiple of n. This relation in **Z** is reflexive, symmetric and transitive.

18. The relation R on the set A = {1, 2, 3} defined as R = {{1, 1), (1, 2), (2, 1), (3, 3)} is reflexive, symmetric and transitive.

ANSWERS

1. (b,b), (c,c), (a,c)

4. Domain of R = {1,2,3,4, ... 20} and Range of R = {1,3,5,7,9, ... 39}. R is neither reflective, nor symmetric and nor transitive.

5. (i) transitive (ii) symmetric (iii) reflexive, symmetric and transitive (iv) transitive.

6. [(2, 5)] = {(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)}

7. C **8.** B **9.** D **10.** B **11.** B

12. A **13.** R = 3,8 , 6,6 , (9,4), (12,2)

14. R = {(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 3), (3, 4), (4, 4), (5, 5)}

15. False **16.** False **17.** False **18.** False

FUNCTION

1. FUNCTION

Let A and B be two non-empty sets and f is a rule which associates each element of A with a unique element of B is called a mapping or function from A to B. If f is a function from A to B, then we write f : A $\to$ B which is read as f is a mapping from A to B.

1.1 DEFINITION (GRAPHICALLY)

If any line parallel to y-axis cuts the graph of the function at most one point, then it denotes a function and if it cuts at more than one point then it is called a relation.

1.2 DIFFERENCE BETWEEN MAPPING AND RELATION

Let A and B be two sets f: A $\to$ B or R : A $\to$ B

Every element belonging to A must have an image in B in the case of Mapping.

Now if R is a relation of being " Husband of " i.e. every man belonging to A is husband of some woman belonging to B.

Above is not the necessary condition. A bachelor or widower in A cannot be husband of some woman in B. All these types of persons in set A will not have any image in B. This is basic difference between a relation and a mapping or function.

1.3 DOMAIN, CO-DOMAIN AND RANGE OF A FUNCTION

Let f : A $\to$ B is a function form A to B, then the set A is called the domain of the function f (denoted by D_f) and the set B is called the Co-domain of the function f (denoted by C_f). The set of all those elements of B which are the images of the elements of set A is called the range of the function f (denoted by R_f).

Domain of f = D_f = {a : a $\in$ A, (a, f (a)) $\in$ f}

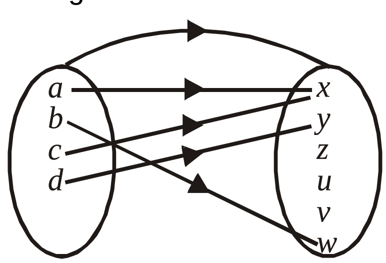
Range of f = R_f = {f (a) : a $\in$ A, f (a) $\in$ B}

It should be noted that the range of f is always a subset of Co-domain B. i.e., $R_f \subseteq C_f$

HOW TO FIND RANGE: First put y = f(x) find x in terms of y. Then find all such y for which x is defined i.e., in the domain. Set of these values of y is the range of f (x).

SOLVED EXAMPLE

1. Find the domain, co-domain and range of the function f which is noted by the figure below.



Sol. It is clear from the figure

Domain of f(D_f) = {a, b, c, d}.

Co-domain of f (C_f) = {x, y, z, u, v, w}.

Range of f (R_f) = {x, y, w}

Also {x, y, w} $\subseteq$ {x, y, z, u, v, w}

i.e., Range $\subseteq$ co-domain.

2. FORMULAS FOR THE DOMAIN OF A FUNCTION

i. Domain of $(f(x) \pm g(x))$ = Domain of f(x) $\cap$ Domain of g (x)

ii. Domain of (f(x).... g(x)) = Domain of f(x) $\cap$ Domain of g (x)

iii. Domain of $\left(\frac{f(x)}{g(x)}\right)$ = Domain of f (x) $\cap$ Domain of g (x) $\cap$ $\{x : g(x) \neq 0\}$

iv. Domain of $\sqrt{f(x)}$ = Domain of f (x) $\cap \{x : f(x) \geq 0\}$

v. Domain of $\log_a f(x)$ = Domain of f(x) $\cap \{x : f(x) > 0\}$

3. VARIOUS TYPES OF FUNCTIONS :

(i) Polynomial Function

If a function f is defined by $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_{n-1} x + a_n$ where n is a **non negative integer** and $a_0, a_1, a_2, ..., a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n.

Note: There are only two polynomial functions, satisfying the relation;

$f(x).f(1/x) = f(x) + f(1/x)$, which are $f(x) = 1 \pm x^n$

Proof : Let $f(x) = a_0 x^n + a_1 x^{n-1} + ... + a_n$, then $f\left(\frac{1}{x}\right) = \frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + ... + a_n$.

Since the relation holds for many values of x,

$\therefore$ Comparing the coefficients of x^n, we get $a_0 a_n = a_0 \Rightarrow a_n = 1$

Similary comparing the coefficients of x^{n-1}, we get $a_0 a_{n-1} + a_1 a_n = a_1$

$\Rightarrow a_{n-1} = 0$, like wise $a_{n-2}, ..., a_1$ are all zero.

Ccomparing the constant terms, we get $a_0^2 + a_1^2 + ... + a_n^2 = 2a_n^2 \quad \Rightarrow \quad a_0 = \pm 1$

(ii) Algebraic Function

y is an algebraic function of x, if it is a function that satisfies an algebraic equation of the form, $P_0(x) y^n + P_1(x) y^{n-1} + ... + P_{n-1}(x) y + P_n(x) = 0$ where n is a positive integer and $P_0(x), P_1(x)$....... are polynomials in x. e.g. $y = |x|$ is an algebraic function, since it satisfies the equation $y^2 - x^2 = 0$.

Note: ☞ All polynomial functions are algebraic but not the converse.

☞ A function that is not algebraic is called **Transcendental Function**.

(iii) Rational Function

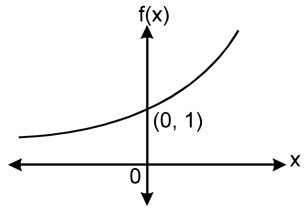
A rational function is a function of the form, $y = f(x) = \frac{g(x)}{h(x)}$, where g (x) & h (x) are polynomials.

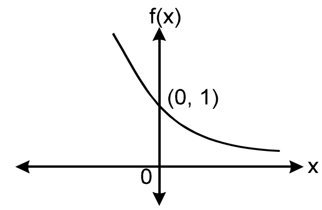
(iv) Exponential Function

A function $f(x) = a^x = e^{x \ln a}$ $(a > 0, a \neq 1, x \in R)$ is called an exponential function. Graph of exponential function can be as follows :

Case - I **For a > 1**

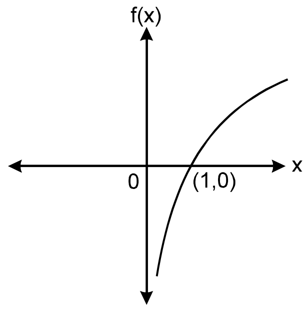
Case - II **For 0 < a < 1**



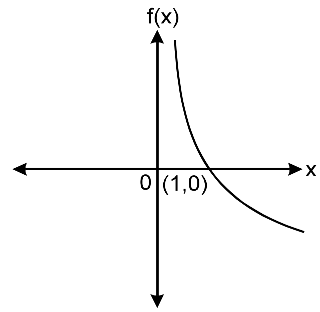


(v) Logarithmic Function : $f(x) = \log_a x$ is called logarithmic function where $a > 0$ and $a \neq 1$ and $x > 0$. Its graph can be as follows

Case- I — **For a > 1**

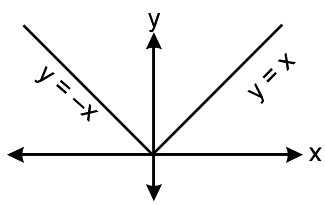


Case- II — **For 0 < a < 1**



(vi) Absolute Value Function / Modulus Function :

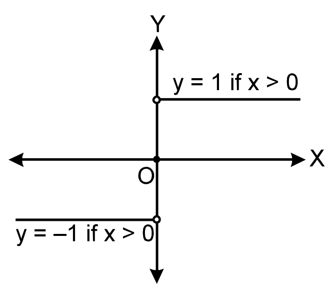
The symbol of modulus function is $f(x) = |x|$ and is defined as: $y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$.



y = |x|

(vii) Signum Function : (Also known as sign(x))

A function f (x) = sgn (x) is defined as follows : $f(x) = \text{sgn}(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$



y = sgn x

It is also written as sgn x = $\begin{cases} \dfrac{|x|}{x}; & x \neq 0 \\ 0; & x = 0 \end{cases}$

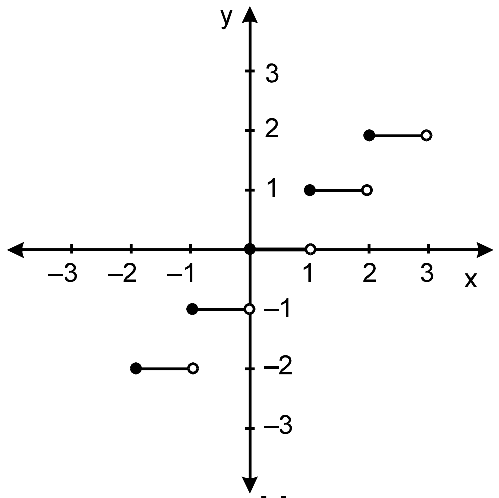
Note: sgn f(x) = $\begin{cases} \dfrac{|f(x)|}{f(x)}; & f(x) \neq 0 \\ 0; & f(x) = 0 \end{cases}$

(viii) Greatest Integer Function or Step Function :

The function y = f (x) = [x] is called the greatest integer function where [x] equals to the greatest integer less than or equal to x. For example :

for $-1 \leq x < 0$; [x] = –1 ; for $0 \leq x < 1$; [x] = 0

for $1 \leq x < 2$; [x] = 1 ; for $2 \leq x < 3$; [x] = 2 and so on.



Properties of greatest integer function :

(a) $x-1 < [x] \le x$

(b) If m is an integer, then $[x \pm m] = [x] \pm m$.

(c) $[-x] = -[x] - 1$

(d) $[x] + [-x] = \begin{cases} 0 & , \text{ if } x \text{ is an integer} \\ -1 & , \text{ if } x \text{ is not an integer} \end{cases}$

(e) $[x] + [y] \le [x+y] \le [x] + [y] + 1$

(f) $[x] \ge k \Rightarrow x \ge k$, where $k \in Z$

(g) $[x] \le k \Rightarrow x < k$, where $k \in Z$

(h) $[x] > k \Rightarrow x \ge k + 1$, where $k \in Z$

(i) $[x] < k \Rightarrow x < k$, where $k \in Z$

(j) $[x + y] = [x] + [y + x - [x]]$ for all $x, y \in R$

(k) $[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + ... + \left[x + \frac{n-1}{n}\right] = [nx]$, $n \in N$.

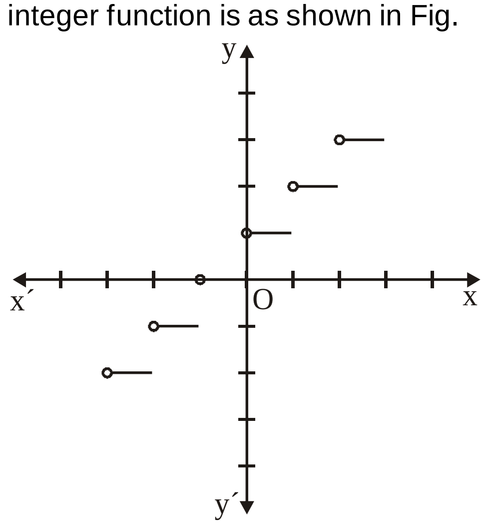
(ix) SMALLEST INTEGER FUNCTION (CEILING FUNCTION): For any real number x, we use the symbol $\lceil x \rceil$ to denote the smallest integer greater than or equal to x.

For example: $\lceil 4.7 \rceil = 5, \lceil -7.2 \rceil = -7 \lceil 5 \rceil = 5, \lceil 0.75 \rceil = 1$ etc.

The function $f : R \to R$ defined by $f(x) = \lceil x \rceil$ for all $x \in R$ *is called the smallest integer function or the ceiling function*. It is also a step function.

We observe that the domain of the smallest integer function is the set R of all real numbers and its range is the set Z of all integers.

The graph of the smallest integer function is as shown in Fig.



(x) PROPERTIES OF SMALLEST INTEGER FUNCTION: Following are some properties of smallest integer function:

(i) $\lceil -n \rceil = -\lceil n \rceil$, where $n \in Z$

(ii) $\lceil -x \rceil = -\lceil x \rceil + 1$, where $n \in R - Z$

(iii) $\lceil x + n \rceil = \lceil x \rceil + n$, where $x \in R - Z$ and $n \in Z$

(iv) $\lceil x \rceil + \lceil -x \rceil = \begin{cases} 1, & \text{if } x \notin Z \\ 0, & \text{if } x \in Z \end{cases}$

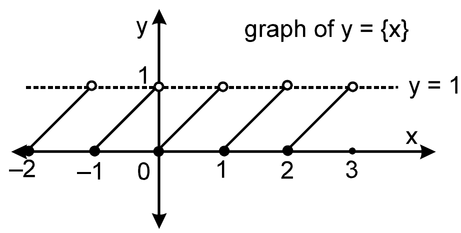
(v) $\lceil x \rceil + \lceil -x \rceil = \begin{cases} 2\lceil x \rceil - 1, & \text{if } x \in Z \\ 2\lceil x \rceil, & \text{if } x \notin Z \end{cases}$

(xi) Fractional Part Function

It is defined as, $y = \{x\} = x - [x]$.

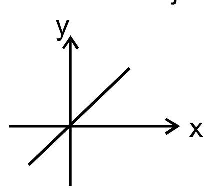
e.g. the fractional part of the number 2.1 is $2.1 - 2 = 0.1$ and $\{-3.7\} = 0.3$.

The period of this function is 1 and graph of this function is as shown.



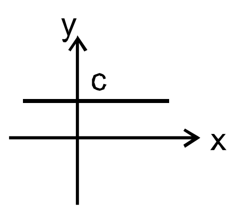
(xii) Identity function

The function $f : A \to A$ defined by, $f(x) = x \ \forall \ x \in A$ is called the identity function on A and is denoted by I_A. It is easy to observe that identity function is a bijection.



(xii) Constant function

A function $f : A \to B$ is said to be a constant function, if every element of A has the same f image in B. Thus $f : A \to B$; $f(x) = c$, $\forall \ x \in A$, $c \in B$ is a constant function.

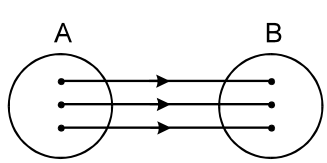


4. DIFFERENT TYPES OF MAPPINGS

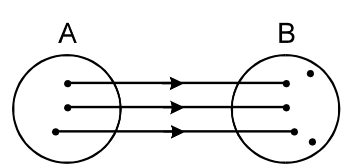
4.1 ONE-ONE MAPPING (INJECTIVE)

The mapping $f : A \to B$ is called one-one mapping or function if different elements in A have different f-images in B. Such a mapping is known as injective mapping also.

One–One Function: Diagrammatically an injective mapping can be shown as



OR



4.2 METHODS TO TEST ONE-ONE:

a. **Analytically:** If $x_1, x_2 \in A$ then $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ or Equivalently $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

b. **Graphically:** If any line parallel to x-axis cuts the graph of the function at most at one point, then the function is one-one.

SOLVED EXAMPLE

1 Prove that the map $f : A \to B$ given by $f(x) = 2x + 5$ is one-one.

Sol. **I Method (Analytically)**

Let $x_1, x_2 \in A$ then $f(x_1) = f(x_2) \Rightarrow 2x_1 + 5 = 2x_2 + 5 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$

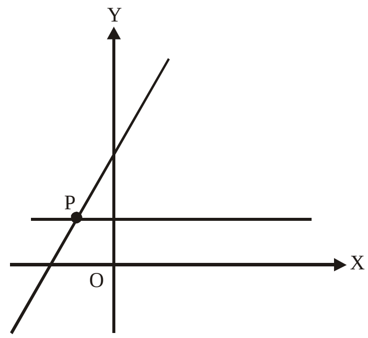
or If $x_1 \neq x_2$

$\Rightarrow 2x_1 \neq 2x_2 \Rightarrow 2x_1 + 5 \neq 2x_2 + 5 \Rightarrow f(x_1) \neq f(x_2)$

Hence f (x) is one-one.

II Method (Graphically)

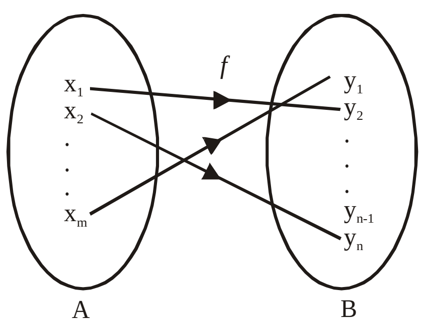
First draw the graph of y = 2x + 5



Since any line parallel to x-axis cuts the graph at one point P hence f (x) is one-one.

4.3 NUMBER OF ONE-ONE MAPPING

Let $f : A \to B$ be a map such that A & B are finite sets having m and n elements respectively, (where n > m).



Let $A = \{x_1, x_2, x_3, ..., x_m\}$ and $B = \{y_1, y_2, y_3, ..., y_n\}$

It is clear for one-one mapping No. of possible images for $x_1 = n$

After x_1 No. of possible images for $x_2 = (n - 1)$

After x_2 No. of possible images for $x_3 = (n - 2)$

After x_{m-1} No. of images for $x_m = (n - m + 1)$.

Hence no. of mappings = $n(n-1)(n-2)....(n-m+1) = \begin{cases} {}^nP_m, & n \geq m \\ 0, & n < m \end{cases}$

SOLVED EXAMPLE

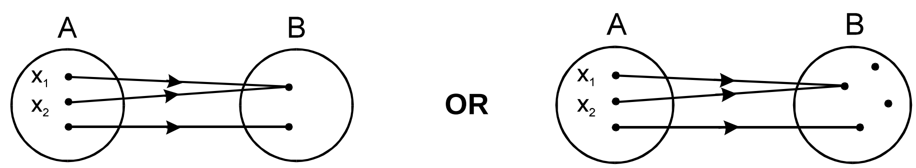
1. A mapping is selected at random from the set of all mappings of the set A = {1, 2, 3, ..., n} into itself. Then find the number of one-one mappings.

Sol. The number of one-one mappings = $^nP_n = n!$

4.4 MANY ONE MAPPING:

The mapping $f : A \rightarrow B$ is called many-one mapping or function if there exist at least two or more elements of A having the same f image in B.

Diagrammatically a many one mapping can be shown as



Note: If a function is one–one, it cannot be many–one and vice versa.

4.5 METHODS TO TEST MANY-ONE

a. **Analytically :** If $x_1, x_2 \in A$ & $f(x_1), f(x_2) \in B$, equate $f(x_1)$ and $f(x_2)$ and if it implies that $x_1 = x_2$, then and only then function is ONE-ONE otherwise MANY-ONE or If $x_1, x_2 \in A$ Then, $f(x_1) = f(x_2) \Rightarrow x_1 \neq x_2$ uniquely then also function is MANY-ONE.

b. **Graphically :** If there exists a straight line parallel to x-axis, which cuts the graph of the function atleast at two points, then the function is MANY-ONE, otherwise ONE-ONE.

SOLVED EXAMPLE

1. Prove that the map $f : A \rightarrow B$ given by $f(x) = x^2 + x + 1 \ \forall x \in R$ is many-one.

Sol. **I Method (Analytically)**

Let $x_1, x_2 \in A$ Then $f(x_1) = f(x_2) \Rightarrow x_1^2 + x_1 + 1 = x_2^2 + x_2 + 1$

$\Rightarrow x_1^2 - x_2^2 + x_1 - x_2 = 0 \Rightarrow (x_1 - x_2)(x_1 + x_2 + 1) = 0$

$\Rightarrow x_1 = x_2$ or $x_1 + x_2 = -1 \because x_1 \neq x_2$ uniquely.

Hence f (x) is many-one.

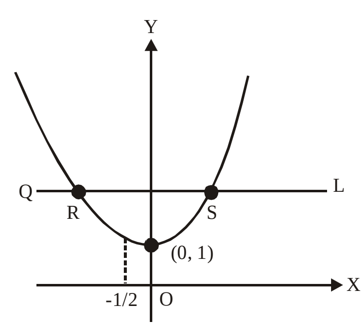
II Method (Graphically)

Let $y = f(x) = x^2 + x + 1 \quad \Rightarrow y = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$ or $\left(x + \frac{1}{2}\right)^2 = \left(y - \frac{3}{4}\right)$

which is a parabola whose vertex is at $\left(-\frac{1}{2}, \frac{3}{4}\right)$

Cuts the y-axis at (0, 1) and neither cuts nor touch the x-axis.

Graph of $y = x^2 + x + 1$ is



Since any line QL parallel to x-axis cuts the graph at two points R and S. Hence f (x) is many-one.

Alternatively: f (1) = 3 and f (-1) = 3 so the function is not one as different elements in A does not have different images in B so function is

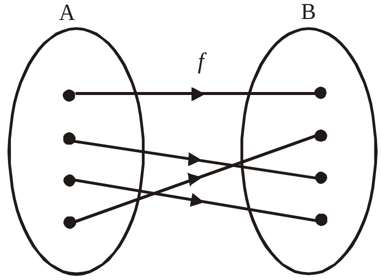
4.6 ONTO (SURJECTIVE) MAPPING

Onto function:

If the function f : A $\rightarrow$ B is such that each element in B (co–domain) must have atleast one pre–image in A, then we say that f is a function of A 'onto' B. Thus f : A$\rightarrow$ B is surjective iff $\forall$ b $\in$ B, there exists some a $\in$ A such that f (a) = b   i.e., Range = Co-domain

Diagrammatically surjective mapping can be shown as

Surjective mapping can be shown as



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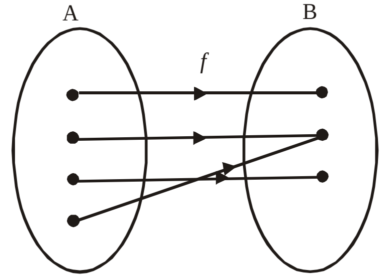


Note: **Every polynomial function f : R$\rightarrow$R of degree odd is ONTO.**

a. Number of Onto (Surjective) Mappings from A to B

Let f : A$\rightarrow$B be a map such that A & B are finite sets having m and n elements respectively such that 1 $\leq$ n $\leq$ m, then number of onto (surjective) mappings from A to B is $r^n - {}^rC_1 (r-1)^n + {}^rC_2 (r-2)^n - ... + (-1)^{r-1}\,{}^rC_{r-1}$

SOLVED EXAMPLE

1. Find the number of surjections from A = {1, 2, 3, ..., n}, n $\geq$ 2 to B = {a, b, c}

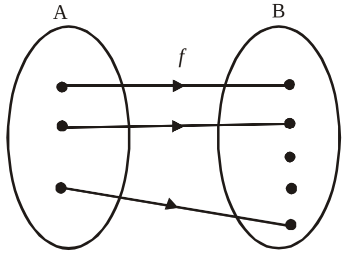
Sol. Hence number of surjections from A to B is $\sum_{r=1}^{3} (-1)^{3-r}\ {}^3C_r r^n$

$= (-1)^2 . {}^3C_1 . (1)^{1/2} + (-1)^1 . {}^3C_2 . (2)^n + (-1)^0 . {}^3C_3 . (3)^n \quad = 3 - 3 (2^n) + (3)^n = 3^n - 3 (2^n - 1)$

4.7 INTO MAPPING

If f : A$\rightarrow$B is such that there exists at least one element in co-domain which is not the image of any element in domain, then f(x) is called into mapping.

Into mapping can be shown as in figure.



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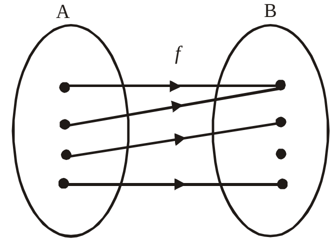


4.8 METHOD TO TEST ONTO OR INTO MAPPING

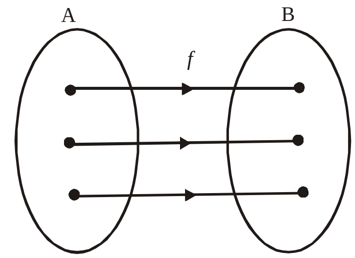
Let f : A$\rightarrow$B be a mapping. Let y be an arbitrary element in B and then y = f(x) where x $\in$A. Then express x in terms of y. Now in x$\in$A $\forall$ y$\in$B then f is onto and if x$\notin$A $\forall$ y$\in$B then f is into.

OR

Find the range of the function. If Range = Codomain then function is onto otherwise it is into.

Note: **For into mapping :** Find an element of B which is not f-image of any element of A.

4.9 ONE-ONE ONTO (INJECTIVE AND SURJECTIVE) OR BIJECTIVE MAPPING



A map $f : A \to B$ is said to be one-one onto or bijective if and only if

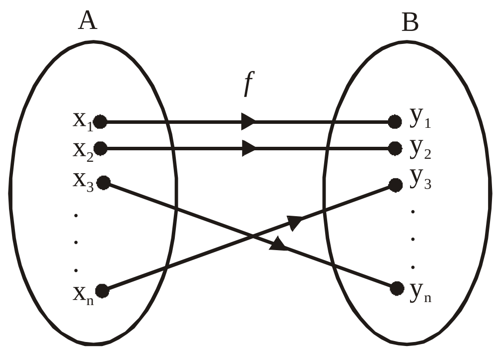
i. It is one-one i.e., $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \ \forall x_1, x_2 \in A$

ii. It is onto i.e., $\forall\ y \in B$, there exists $x \in A$ such that $f(x) = y$.

One-one onto mapping can be shown as

Number of one-one onto mappings or bijections:

Let $f : A \to B$ be a map such that A and B are finite sets having the same number of elements. If A has n elements then B has also n elements.



Let $A = \{x_1, x_2, x_3, ..., x_n\}$ and $B = \{y_1, y_2, y_3, ..., y_n\}$ It is clear for bijective mapping

No. of possible images for $x_1 = n$.

After x_1 No. of possible images for $x_2 = (n-1)$

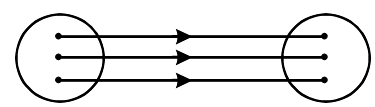
After x_2 No. of possible images for $x_3 = (n-2)$

After x_{n-1} No. of possible images for $x_n = 1$

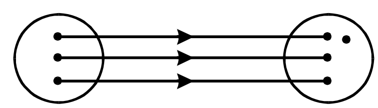
Hence total number of bijective mappings = $n(n-1)(n-2)(n-1) \ldots 2.1 = n!$

4.10 A FUNCTION CAN BE ONE OF THESE FOUR TYPES

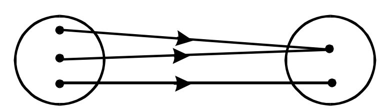
(a) one–one onto (injective & surjective)



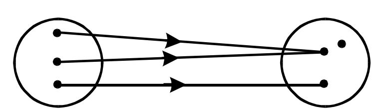
(b) one–one into (injective but not surjective)



(c) many–one onto (surjective but not injective)



(d) many–one into (neither surjective nor injective)



Note:

(i) If f is both injective & surjective, then it is called a **bijective** mapping. The bijective functions are also named as invertible, non singular or biuniform functions.

(ii) If a set A contains 'n' distinct elements then the number of different functions defined from $A \to A$ is n^n and out of which $n!$ are one one.

OBJECTIVE TYPE EXERCISE

1. $f : X \to Y$ is onto, if and only if

a. Range of f = Y b. Range of $f \neq$ Y c. Range of f < Y d. Range of $f \geq$ Y

2. A function $f : X \to Y$ is said to be bijective, if f is

a. one-one only b. onto only c. both one-one & onto d. either one-one or onto

3. Consider the following statements

Statement I : An onto function $f : \{1, 2, 3\} \to \{1, 2, 3\}$ is always one-one.

Statement II : A one-one function $f : \{1, 2, 3\} \to \{1, 2, 3\}$ must be onto.

Choose the correct option

a. Only I is true b. Only II is true c. Both I and II are true d. Neither I nor II is true

4. Consider the following statements :

Statement I : The function $f : R_* \to R_*$ defined by $f(x) = \frac{1}{x}$ is one-one and onto, where R_* is the set of all non-zero real numbers.

Statement II : The function $g : N \to R_*$ defined by $f(x) = \frac{1}{x}$ is one-one and onto.

Choose the correct option.

a. Only I is true b. Only II is true c. Both I and II are true d. Neither I nor II is true

5. Let A = {1, 2, 3} and B = {4, 5, 6, 7} and let f = {(1, 4), (2, 5), (3, 6)} be a function from A to B. Then, f is

a. one-one b. onto c. many-one d. bijective

6. The greatest integer function $f : R \to R$, given by $f(x) = [x]$ is

a. one-one b. onto

c. both one-one and onto d. neither one-one nor onto

7. Which of the following function is one-one?

a. $f(x) = \sin x,\ x \in [-\pi, \pi]$ b. $f(x) = \sin,\ x \in \left[-\frac{3\pi}{2}, -\frac{\pi}{4}\right]$

c. $f(x) = \cos x,\ x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ d. $f(x) = \cos x,\ x \in \left[\pi, \frac{3\pi}{2}\right]$

8. Let n(A) = 4 and n(B) = 6. Then, the number of one-one functions from A to B is

a. 20 b. 60 c. 120 d. 360

9. The value of parameter α, for which the function $f : R \to R$ given by $f(x) = 1 + \alpha x$, $\alpha \neq 0$ is the inverse of itse,f is

a. –2 b. –1 c. 1 d. 2

10. Let the set A has 3 elements and B has 4 elements. Then, the number of injections that can be defined from A to B, is

a. 144 b. 12 c. 24 d. 64

ANSWERS

1. a	**2.** c	**3.** c	**4.** a	**5.** a
6. d	**7.** d	**8. d**	**9.** b	**10.** a

PREVIOUS YEARS BOARD QUESTIONS

1. If A = {1, 2, 3}, B = {4, 5, 6, 7} and f = {(1, 4), (2, 5), (3, 6)} is a function from A to B. State whether f isone-one or not. **[A.I 2011]**

2. State whether the function f : N $\rightarrow$ N given by f(x) = 5x is injective, surjective or both. **[A.I 2009C]**

3. Show that f : N $\rightarrow$ N, given by f(x)= $\begin{cases} x+1, \text{ if } x \text{ is odd} \\ x-1, \text{ if } x \text{ is even} \end{cases}$ is bijective (both one-one and onto). **[A.I 2012]**

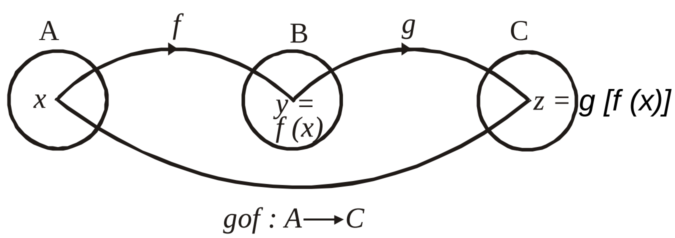
4. If f : N$\rightarrow$N is defined by f(n) = $\begin{cases} \frac{n+1}{2}, \text{ if } n \text{ is odd} \\ \frac{n}{2}, \quad \text{ if } n \text{ is even} \end{cases}$ for all $n \in N$. Find whether the function f is bijective. **[A.I 2009]**

ANSWERS

1. It is one-one **2.** Injective but not surjective **4.** Many one onto.

5. COMPOSITION OF FUNCTIONS

Let A, B and C be three non-empty sets. Let f : A $\rightarrow$ B and g : B $\rightarrow$ C be two functions then gof : A $\rightarrow$ C. This function is called the product or composite of f and g, given by gof(x) = g[f(x)]



Thus the image of every x $\in$ A under the function *gof* is the g-image of the f-image of x.

Notes:

1. The gof is defined only if $\forall$ x $\in$ A, f (x) is an element of the domain of g so that we can take its g-image.

2. The range of *f* must be a subset of the domain of g in *gof*.

3. i. (fog) x = f [g (x)] ii. (fof) x = f [f(x)] iii. (gog) x = g [g (x)] iv. (fg) x = f (x) . g (x)

v. (f $\pm$ g) x = f (x) $\pm$ g (x) vi. $\left(\frac{f}{g}\right)x = \frac{f(x)}{g(x)}$; g (x) $\neq$ 0

5.1 PROPERTIES OF COMPOSITION OF FUNCTIONS

1. The composition of functions is not commutative i.e., fog $\neq$ gof

2. The composition of functions is associative.

i.e., if h : A $\rightarrow$ B, g : B $\rightarrow$ C and f : C $\rightarrow$ D be three functions, then (fog) oh = fo (goh)

3. Let f : A $\rightarrow$ B, g : B $\rightarrow$ C be two functions, then

i. f and g are injective $\Rightarrow$ gof is injective. ii. f and g are surjective $\Rightarrow$ gof is surjective

iii. f and g are bijective $\Rightarrow$ gof is bijective

4. An injective mapping from a finite set to itself is bijective.

5. The composition of any function with the identity function is the function itself.

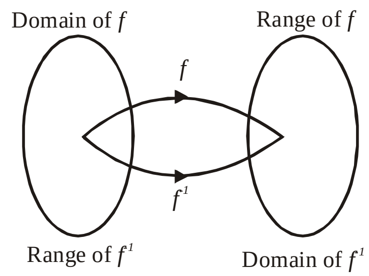
i.e., f : A $\rightarrow$ B then foI_A = $I_B$of = f Where I_A and I_B are the Identity functions of A and B respectively.

5.2 INVERSE OF A FUNCTION

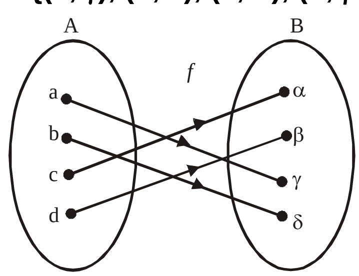
Let $f : A \to B$ be a function defined by $y = f(x)$ such that f is both one-one (injective) and onto (surjective) (i.e., bijective), then there exists a unique function $g : B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x \ \forall\ x \in A$ and $y \in B$ then g is said to be inverse of f.

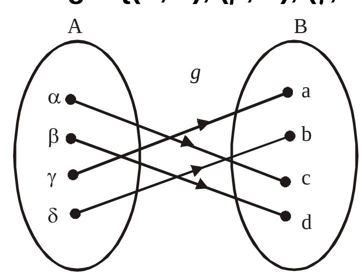
Thus $g = f^{-1} : B \to A = \{(f(x), x) : (x, f(x)) \in f\}$

If f and g are inverse to each other then $(fog)(x) = (gof)(x) = x$ i.e., $f\{g(x)\} = g\{f(x)\} = x$



Consider: $\quad$ **f = {(a, γ), (b, δ), (c, α), (d, β)}** $\quad$ **g = {(α, c), (β, d), (γ, a), (δ, b)]**



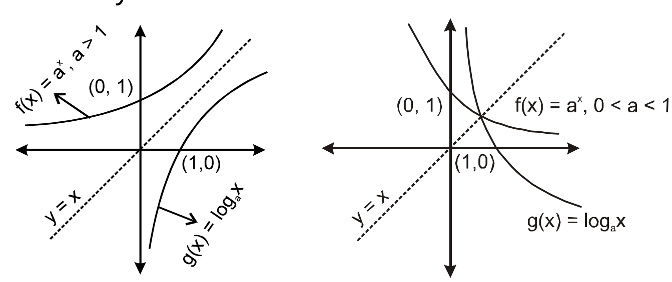


Here g is called the inverse of f i.e., $g = f^{-1}$

5.3 PROPERTIES OF INVERSE FUNCTION

(a) The graphs of f & g are the mirror images of each other in the line y = x.

For example $f(x) = a^x$ and $g(x) = \log_a x$ are inverse of each other, and their graphs are mirror images of each other on the line y = x as shown below.



(b) Normally points of intersection of f and f^{-1} lie on the straight line y = x. However it must be noted that f(x) and $f^{-1}(x)$ may intersect otherwise also. e.g $f(x) = 1/x$

(c) In general fog(x) and gof(x) are not equal. But if f and g are inverse of each other, then gof = fog. fog(x) and gof(x) can be equal even if f and g are not inverses of each other. e.g. $f(x) = x + 1$, $g(x) = x + 2$. However if $fog(x) = gof(x) = x$, then $g(x) = f^{-1}(x)$

(d) If f & g are two bijections $f : A \to B$, $g : B \to C$ then the inverse of gof exists and $(gof)^{-1} = f^{-1} \text{ o } g^{-1}$.

How to find f^{-1}

First let $y = f(x) \Rightarrow x = f^{-1}(y)$ ...(1)

and solve the equation $y = f(x)$

For x in terms of y $\Rightarrow x = \phi(y)$ (say) ...(2)

From (1) and (2), we get, $f^{-1}(y) = \phi(y)$

Replacing y be x, we get, $f^{-1}(x) = \phi(x)$ which is the required inverse of f (x).

SOLVED EXAMPLE

1. Let f : R $\to$ R be defined as f(x) = 10x + 7. Find the function g : R $\to$ R such that gof = fog = I_R.

Sol. gof = fog = $I_R \Rightarrow$ g(x) = f^{-1} (x).

Now let f(x) = y $\Rightarrow$ x = f^{-1} (y) ...... (1)

$\Rightarrow$ y = 10 x + 7 or x = $\frac{y-7}{10}$ $\Rightarrow$ f^{-1}(y) = $\frac{y-7}{10}$ [using ..1]

$\Rightarrow$ f^{-1}(x) = $\frac{x-7}{10}$ $\Rightarrow$ g(x) = $\frac{x-7}{10}$

OBJECTIVE TYPE EXERCISE

1. Consider the following statements

I. If f : A $\to$ B and g : B $\to$ C are one-one, then gof : A $\to$ C is also one-one.

II. If f : A $\to$ B and g : B $\to$ C are onto, then gof : A $\to$ C is not onto.

Choose the correct option.

a. Only I is the b. Only II is true c.Both I and II are true d. Neither I nor II is true

2. If the function gof is defined and is one-one, then

a. neither f nor g is one-one b. f and g both are necessarily one-one

c. g must be one-one d. None of the above

3. If the function gof is defined and onto, then

a. neither f nor g is onto b. f and g both are necessarily onto

c. f must be onto d. None of the above

4. Which of the following options is correct?

a. gof is one $\Rightarrow$ g is one-one b. gof is one-one $\Rightarrow$ f is one-one

c. gof is onto $\Rightarrow$ g is not onto d. gof is onto $\Rightarrow$ f is onto

5. If f : X $\to$ Y is a function such that there exists a function g : Y $\to$ X such that gof = I_X and fog = I_Y, then f must be

a. one-one b. onto c. one-one and onto d. None of these

6. Let f : {1, 3, 4} $\to$ {1, 2, 5} and g: {1, 2, 5} $\to$ {1, 3} be given by f = {(1, 2), (3, 5), (4, 1)} and g = {(1, 3), (2, 3), (5, 1)}. Then, gof is

a. {(1, 3), (3, 1), (4, 3)} b. {(1, 3), (4, 3)}

c. {(3, 1), (4, 3)} d. {(3, 1), (1, 3)}

7. If f(x) = |x| and g(x) = |5x – 2|, then

a. gof(x) = |5x – 2| b. gof = |5|x| –2| c. fog(x) = |5|x| –2| d. fog = |5x + 2|

8. If f(x) = $8x^3$ and g(x) = $x^{1/3}$, then

a. fog(x) = 2x b. fog = 8x c. gof(x) = $2x^{1/3}$ d. gof(x) = $x^{1/3}$

9. If f(x) = $\frac{\alpha x}{x+1}$, $x \neq -1$, then for what value of α, f{f(x)} = x?

a. $\sqrt{2}$ b. $-\sqrt{2}$ c. –1 d. 2

ANSWERS

1. a **2.** d **3.** d **4.** b **5.** c **6.** a **7.** b **8.** b **9.** c

ASSIGNMENT

1. Which of the following represents function $f : R \to R$? **a.** $y = x^2$ **b.** $y^2 = x$
2. Check the injectivity and surjectivity of the following functions:
   i. $f : \mathbf{N} \to \mathbf{N}$ defined by $f(x) = x^2$ **ii.** $f : \mathbf{Z} \to \mathbf{Z}$ defined by $f(x) = x^2$
   iii. $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2$ **iv.** $f : \mathbf{N} \to \mathbf{N}$ defined by $f(x) = x^3$
   v. $f : \mathbf{Z} \to \mathbf{Z}$ defined by $f(x) = x^3$ **vi.** $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^3$
3. Let A be the set of all 50 students of class XII in a central school.
   Let $f : A \to N$ be a function defined by $f(x)$ = Roll number of student x. Show that f is one-one but not onto.
4. Let $f : N - \{1\} \to N$ be defined by f(n) = the highest prime factor of n.Show that f is neither one-one nor onto.
5. Show that the function $f : N \to N$, given by $f(x) = 2x$, is one-one but not onto.
6. Show that the function $f : R_0 \to R_0$, define as $f(x) = \frac{1}{x}$, is one-one onto, where R_0 is the set of all non-zero real numbers. Is the result true, if the domain R_0 is replaced by N with co-domain being same as R_0?
7. Prove that function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 3x^2 - 2$ is many-one into.
8. Show that the function $f : R \to R$ defined by $f(x) = 3x^3 + 5$ for all $x \in R$ is a bijection.
9. Show that the function $f : R \to R$ given by $f(x) = x^3 + x$ is a bijection.
10. Let $A = R - \{2\}$ and $B = R - \{1\}$. If $f : A \to B$ is a mapping defined by $f(x) = \frac{x-1}{x-2}$, show that f is bijective.
11. **i.** Let **Z** be the set of all integers. Show that the function $f : \mathbf{Z} \to \mathbf{Z}$ defined by $f(x) = |x|$ is neither one-one nor onto.
    ii. What would be the answer if $f : \mathbf{N} \to \mathbf{N}$. **iii.** What would be the answer if $f : \mathbf{R} \to \mathbf{R}$.
12. Let $A = \{x \in R : -1 \le x \le 1\} = B$. Show that $f : A \to B$ given by $f(x) = x\,|x|$ is a bijection.
13. Show that the greatest integer function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = [x]$ is neither one-one nor onto. What would be the answer if $f : \mathbf{Z} \to \mathbf{Z}$.
14. Find whether $f : N \to N$ defined by $f(n) = \begin{cases} \frac{n+1}{2}, & if\ n\ is\ \text{odd} \\ \frac{n}{2}, & if\ n\ is\ \text{even} \end{cases}$ is many one onto function.
15. Show that the function $f : N \to N$ given by $f(n) = n - (-1)^n$ for all $n \in N$ is a bijection.
16. Show that the function $f : R \to \{x \in R : -1 < x < 1\}$ defined by $f(x) = \frac{x}{1+|x|}$, $x \in R$ is one one & onto function.
17. Let A and B be two sets. Show that $f : A \times B \to B \times A$ defined by $f(a,b) = (b, a)$ is a bijection.
18. Consider a function $f : \left[0, \frac{\pi}{2}\right] \to \mathbf{R}$ given by $f(x) = \sin x$ and a function $g : \left[0, \frac{\pi}{2}\right] \to \mathbf{R}$ given by $g(x) = \cos x$. Show that both f and g are one-one but $f + g$ is not one-one.
19. Find *fog* and *gof*:
    i. If $f : R \to R$; $f(x) = x^2$ and $g : R \to R$; $g(x) = 2x + 1$. **ii.** Let $f : R \to R$; $f(x) = \sin x$ and $g : R \to R$; $g(x) = x^2$
    iii. If $f : R \to R$, $f(x) = x^2 + 2$ and $g : R \to R$, $g(x) = \frac{x}{x-1}$. **iv.** If $f, g : R \to R$ $f(x) = |x|$ & $g(x) = |5x - 2|$

20. If $f(x) = e^x$ and $g(x) = \log_e x$ $(x > 0)$, find fog and gof. Is fog = gof.

21. $f : R \to R$ be given by $f(x) = (3 - x^3)^{1/3}$, then find fof (x) and gog (x).

22. **i.** Let $f : \{1, 3, 4\} \to \{1, 2, 5\}$ and $g : \{1, 2, 5\} \to \{1, 3\}$ be given by $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(1, 3), (2, 3), (5, 1)\}$. Write down gof and fog if exsist..

ii. If f, g are the functions, given by $f = \{(1, 2), (2, 3), (3, 7), (4, 6)\}$, $g = \{(0, 4), (1, 2), (2, 1)\}$ find gof and fog if exsist.

iii. Let $f : \{2, 3, 4, 5\} \to \{3, 4, 5, 9\}$ and $g: \{3, 4, 5, 9\} \to \{7, 11, 15\}$ be functions defined as $f(2) = 3$, $f(3) = 4$, $f(4) = f(5) = 5$ & $g(3) = g(4) = 7$ and $g(5) = g(9) = 11$. Find gof and fog if exist.

23. If $f : R \to R$ be given by $f(x)=\sin^2 x+\sin^2 (x+\pi/3)+\cos x.\cos (x+\pi/3)$ for all $x \in R$, and $g : R \to R$ be such that $g(5/4) = 1$, then prove that $gof : R \to R$ is a constant function.

24. Let $f(x) = \begin{cases} 2x-3, & x>2 \\ 1+x, & x \le 2 \end{cases}$ and $g(x) = \begin{cases} x^2, & x \le 3 \\ 4-x, & x>3 \end{cases}$ be functions from R to R. Find (fog) (2).

25. Let f (x) = [x] and g(x) = | x |. Find **i.** $(gof)\left(\frac{-5}{3}\right)-(fog)\left(\frac{-5}{3}\right)$ **ii.** $(gof)\left(\frac{5}{3}\right)-(fog)\left(\frac{5}{3}\right)$

26. If $f : R-\left\{\frac{7}{5}\right\} \to R-\left\{\frac{3}{5}\right\}$ be defined as *f(x)* = $\frac{3x+4}{5x-7}$ and $g : R-\left\{\frac{3}{5}\right\} \to R-\left\{\frac{7}{5}\right\}$ be defined as $g:(x) = \frac{7x+4}{5x-3}$. Show that *gof* = I_A and *fog* = I_B, where $B = R - \left\{\frac{3}{5}\right\}$ and $A = R - \left\{\frac{7}{5}\right\}$.

27. Let $f : Z \to Z$ be defined by $f(n) = 3n$ for all $n \in Z$ and $g : Z \to Z$ be defined by

$g(n) = \begin{cases} \frac{n}{3}, \text{ if } n \text{ is a multiple of } 3 \\ 0, \text{ if } n \text{ is not a multiple of } 3 \end{cases}$ for all $n \in Z$. Show that *gof* = I_Z and *fog* $\neq I_Z$.

28. Let $f : R \to R$ be a function given by $f(x) = ax+b$ for all $x \in R$. Find the constant *a* and *b* such that *fof* = I_R.

29. Let $f : Z \to Z$ be defined by $f(x) = x + 2$. If $g : Z \to Z$ such that gof = I_Z, find g (x).

30. Let f, g & h be functions from R to R. Show:**a.** (f + g) o h = f o h + g o h **b.** (f . g) o h = (f o h) .(g o h)

31. Consider $f : N \to N$, $g : N \to N$ and $h : N \to N$ defined as $f(x) = 2x$, $g(y) = 3y + 4$ and $h(z) = \sin z$, $\forall$ x, y and z in N. Show that ho(gof) = (hog)of.

32. State with reason whether following functions have inverse

i. $f : \{1, 2, 3, 4\} \to \{10\}$ with $f = \{(1, 10), (2, 10), (3, 10), (4, 10)\}$

ii. $g : \{5, 6, 7, 8\} \to \{1, 2, 3, 4\}$ with $g = \{(5, 4), (6, 3), (7, 4), (8, 2)\}$

iii. $h : \{2, 3, 4, 5\} \to \{7, 9, 11, 13\}$ with $h = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$

iv. Let S = {1, 2, 3}. and $I: S \to S$ $I = \{(1, 2), (2, 1), (3, 1)\}$

33. Consider $f : \{1, 2, 3\} \to \{a, b, c\}$ given by $f(1) = a$, $f(2) = b$ & $f(3) = c$. Find f^{-1} and show that $(f^{-1})^{-1} = f$.

34. If $f : R \to R$ is defined by $f(x) = 2x + 7$. Prove that *f* is a bijection and find the inverse of *f*.

35. Let $f : R \to R$ be defined as $f(x) = x^2 + 1$. If f is invertible then find : **i.** $f^{-1}(-5)$ **ii.** $f^{-1}(26)$ **iii.** $f^{-1}\{10, 37\}$.

36. If A = {1, 2, 3, 4}, B = {2, 4, 6, 8} and $f : A \to B$ is given by *f(x)* = 2x, such that f is invertible, then write *f* and f^{-1} as a set of ordered pairs.

37. Show that $f : R-\{0\} \to R-\{0\}$ given by $f(x) = \frac{3}{x}$ is invertible and it is inverse of itself.

38. Show that $f : R-\{-1\} \to R-\{1\}$ given by $f(x) = \frac{x}{x+1}$ is invertible. Also, find f^{-1}

39. Show that $f:[-1,1]\to R$, given by $f(x)=\frac{x}{x+2}$ is one-one. Find the inverse of the function f : [–1, 1] $\to$ Range f.

40. Let f : N$\to$R be a function defined a f(x) = $4x^2+12x+15$. Show f : N$\to$Range (f) is invertible. Find inverse of '*f*'.

41. Let *f* : $R_+\to[-5,\infty)$ be a function defined as *f*(x) = $9x^2+6x-5$. Show that *f* is invertible. Find the inverse of *f*.

42. If the function *f* : [1, ∞) $\to$ [1, ∞) defined by *f*(*x*) = $2^{x(x-1)}$ is invertible, find $f^{-1}(x)$.

43. Let *f* : N$\cup$\{0\}$\to$N $\cup$ \{0\} be defined by $f(n)=\begin{cases} n+1, \text{if } n \text{ is even} \\ n-1, \text{if } n \text{ is odd}\end{cases}$ Show that *f* is invertible and $f=f^{-1}$.

44. Let A = {1, 2, 3, 4}; B = {3, 5, 7, 9}; C = {7, 23, 47, 79} & f : A$\to$B, g : B$\to$C be defined as f(x) = 2x + 1 and g(x) = x^2-2. Express $(gof)^{-1}$ and $f^{-1}og^{-1}$ as the sets of ordered pairs and prove: $(gof)^{-1}=f^{-1}og^{-1}$.

45. Consider *f* : {1, 2, 3} $\to$ {a, b, c}, g: {a, b, c} $\to$ {apple, ball, cat} defined as *f*(1) = a, *f*(2) = b, *f*(3) = c, *g*(a)= apple, *g*(b) = ball and *g*(c) = cat. Show that f, g, gof are invertible. Also find f^{-1}, g^{-1} and $(gof)^{-1}$ and show that $(gof)^{-1}=f^{-1}og^{-1}$.

46. Find the value of parameter a for which the function f(x) = 1 + ax, a $\neq$ 0 is the inverse of itself.

47. Let A = {– 1, 0, 1, 2}, B = {– 4,– 2, 0, 2} and f, g **:** A$\to$B be functions defined by

f(x) = x^2-x, and g(x) = $2\left|x-\frac{1}{2}\right|-1$. Prove that f = g.

ANSWERS

1. i. yes **ii.** no **2. i.** Injective. **ii.** Neither. **iii.**Neither. **iv.** Injective. **v.** Injective **vi.** Both

6. No **11. i.** one – one and onto **ii.** one – one and onto **iii.** one – one and onto

13. one – one and onto **14.** Yes

19. i. fog (x) = $4x^2+4x+1$, gof (x) = $2x^2+1$ **ii.** fog (x) = $\sin x^2$, gof (x) = $\sin^2 x$

iii. fog (x) = $\frac{3x^2-4x+2}{(x-1)^2}$, gof (x) = $\frac{x^2+2}{x^2+1}$ **iv.** fog(x) = |5x –2|, gof (x) = |5|x | –2 |

20. fog(x) = gof(x) = x, No**21.** x

22. i. gof (x) = {(1, 3), (3, 1), (4, 3)} **ii.** fog (x) = {(0, 6), (1, 3) (2,2)} **iii.** gof = {(2, 7), (3, 7), (4, 11), (5, 11)}

24. 5 **25. i.** 1 **ii.** 0 **28.** a = 1, b = 0 or a = –1 and b $\in$ R

29. g(x) = x –2

32. i. No, f is many one **ii.** No, g is many one **iii.** Yes, h is bijective **iv.** No, l is many one

34. $f^{-1}(x)=\frac{x-7}{2}$ **35. i.** ϕ **ii.** {–5, 5} **iii.** {3, –3, 6, –6} **36.** {(2, 1), (4, 2), (6, 3), (8, 4)}

38. $f^{-1}(x)=\frac{x}{1-x}$ **39.** $f^{-1}(x)=\frac{2x}{1-x}$ **40.** $f^{-1}(x)=\frac{-3+\sqrt{x-6}}{2}$.

41. $f^{-1}(x)=\left(\frac{\sqrt{x+6}-1}{3}\right)$ **42.** $f^{-1}(x)=\frac{1+\sqrt{1+4\log_2 x}}{2}$ **46.** $\alpha=-1$

PREVIOUS YEARS BOARD QUESTIONS

1. If f : {1, 3, 4} $\rightarrow$ {1, 2, 5} and g : {1, 2, 5} $\rightarrow$ {1, 3} given by f = {(1, 2), (3, 5), (4, 1)} and g = {(1, 3), (2, 3), (5,1)}. Write down gof. **[A.I 2014 C]**

2. If f : R $\rightarrow$ R is defined by f (x) = 3x + 2, then define f[f(x)]. **[Delhi 2010]**

3. If f : R $\rightarrow$ R is defined by f (x) = $(3 - x^3)^{1/3}$, then find fof(x). **[A.I 2010]**

4. If f is an invertible function, defined as f(x) $\frac{3x-4}{5}$, then write $f^{-1}(x)$. **[Foreign 2010]**

5. If f : W $\rightarrow$ W, is defined as f(x) = x – 1, if x is odd and f(x) = x + 1, if x is even. Show that f is invertible. Find the inverse of f, where W is the set of all whole numbers. **[Foreign 2014]**

6. If f : R $\rightarrow$ R are two functions defined as f(x) = |x| + x and g(x) = |x| – x, $\forall$ x $\in$ R, Then, find fog & gof. **[A.I 14C]**

7. If A = R – {3} and B = R – {1}.Consider the function f : A $\rightarrow$ B defined by f(x) = $\frac{x-2}{x-3}$ for $f^{-1}(x)$. **[Delhi 2014C]**

8. If A = R – {2} and B = R – {1}. If f : A $\rightarrow$ B is a function defined by f(x) = $\frac{x-1}{x-2}$, then show that f is one-one and onto. Hence, find f^{-1}. **[Delhi 2013C]**

9. Consider f : $R_+ \rightarrow [4, \infty]$ given by f(x) = $x^2 + 4$. Show that f is invertible with the inverse f^{-1} of f given by $f^{-1}(y) = \sqrt{y-4}$, where R_+ is the set of all non-negative real numbers. **[A.I 2013]**

10. If f : R $\rightarrow$ R is defined as f(x) = 10x + 7. Find the function g : R $\rightarrow$ R, such that gof = fog = I_R. **[A.I 2011]**

11. Consider f : $R_+ \rightarrow [-5,\infty]$ given by f(x) = $9x^2+6x-5$, show that f is invertible with $f^{-1}(y) = \left(\frac{\sqrt{y+6}-1}{3}\right)$. **[Foreign 2010]**

12. If the function f : R $\rightarrow$ R is given by f(x) = $\frac{x+3}{3}$ and g : R $\rightarrow$ R is given by g(x) = 2x – 3, then find

(i) fog and (ii) gof. Is f^{-1} = g? **[Delhi 2009]**

ANSWERS

1. {(1, 3), (3, 1), (4, 3)} **2.** 9x + 8 **3.** x **4.** $\frac{5x+4}{3}$

5. $f^{-1} = f$ **6.** $fog(x) = \begin{cases} 0, & x > 0 \\ -4x, & x < 0 \end{cases}$ and gof(x) = 0 **7.** $\frac{3x-2}{x-1}$

8. $\frac{2x-1}{x-1}$ **10.** $g(x) = \frac{x-7}{10}$ **12.** (i) $\frac{2x}{3}$ (ii) $\frac{2x-3}{3}$; No

EXEMPLAR PROBLEMS

SHORT ANSWER

1. Let D be the domain of the real valued function *f* defined by $f(x) = \sqrt{25-x^2}$. Then, write D.

2. Let *f*, *g* : **R** $\rightarrow$ **R** be defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$, $\forall x \in$ **R**, respectively. Then, find *gof*.

3. Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) = 2x - 3$ $\forall$ $x \in$ R. write f^{-1}.

4. If A = {a, b, c, d} and the function f = {(a, b), (b, d), (c, a), (d, c)}, write f^{-1}.

5. If $f : \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = x^2 - 3x + 2$, write $f\{f(x)\}$.

6. Is g = {(1, 1), (2, 3), (3, 5), (4, 7)} a function? If g is described by $g(x) = \alpha x + \beta$, then what value should be assigned to α and β.

7. Are the following set of ordered pairs functions? If so, examine whether the mapping is injective or surjective.

(i) {(x, y): x is a person, y is the mother of x}. (ii){(a, b): a is a person, b is an ancestor of a}.

8. If the mappings f and g are given by f = {(1, 2), (3, 5), (4, 1)} & g = {(2, 3), (5, 1), (1, 3)}, write fog.

9. Let **C** be the set of complex numbers. Prove that the mapping f : $\mathbf{C} \to \mathbf{R}$ given by $f(z) = |z|$, $\forall$ $z \in \mathbf{C}$, is neither one-one nor onto.

10. Let the function $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \cos x$, $\forall$ $x \in \mathbf{R}$. Show that f is neither one-one nor onto.

11. Let X = {1, 2, 3}and Y = {4, 5}. Find whether the following subsets of X ,Y are functions from X to Y or not.

(i) f = {(1, 4), (1, 5), (2, 4), (3, 5)} (ii) g = {(1, 4), (2, 4), (3, 4)}

(iii) h = {(1,4), (2, 5), (3, 5)} (iv) k = {(1,4), (2, 5)}.

12. If functions f : A $\to$ B and g : B $\to$ A satisfy $g \, o \, f = I_A$, then show that f is one one and g is onto.

13. Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) = \dfrac{1}{2 - \cos x}$ $\forall$ $x \in \mathbf{R}$.Then, find the range of f.

<u>LONG ANSWER</u>

14. Given A = {2, 3, 4}, B = {2, 5, 6, 7}. Construct an example of each of the following:

(a) an injective mapping from A to B (b) a mapping from A to B which is not injective

(c) a mapping from B to A.

15. Give an example of a map

(i) which is one-one but not onto (ii) which is not one-one but onto

(iii) which is neither one-one nor onto.

16. Let A = **R** – {3}, B = **R** – {1}. Let f : A $\to$ B be defined by $f(x) = \dfrac{x-2}{x-3}$, $\forall$ $x \in$ A. Then show that f is bijective.

17. Let A = [–1, 1]. Then, discuss whether the following functions defined on A are one-one, onto or bijective:

(i) $f(x) = \dfrac{x}{2}$ (ii) $g(x = |x|$ (iii) $h\ (x)\ x\ |x|$ (iv) $k(x) = x_2$.

18. Using the definition, prove that the function f : A $\to$ B is invertible if and only if f is both one-one and onto.

19. Functions f, g : **R** × **R** are defined, respectively, by $f(x) = x^2 + 3x + 1$, $g(x) = 2x - 3$, find

(i) fog (ii) gof (iii) fof (iv) $g \, o \, g$

<u>OBJECTIVE TYPE QUESTIONS</u>

Choose the correct answer out of the given four options.

20. The function $f(x) = {}^{7-x}P_{x-3}$ is

(A) one-one (B) many one (C) onto (D) one-one and onto both

21. If the set A contains 5 elements and the set B contains 6 elements, then the number of one-one and onto mappings from A to B is

(A) 720 (B) 120 (C) 0 (D) none of these

22. Let A = {1, 2, 3, ...n} and B = {a, b}. Then the number of surjections from A into B is

(A) ${}^{n}P_2$ (B) $2^n - 2$ (C) $2^n - 1$ (D) None of these

23. Let f : **R** → **R** be defined by $f(x) = \frac{1}{x}$, $\forall x \in$ **R**. Then f is

(A) one-one (B) onto (C) bijective (D) f is not defined

24. Let f : **R** → **R** be defined by $f(x) = 3x^2 - 5$ and g : **R** → **R** by $g(x) = \frac{x}{x^2+1}$. Then gof is

(A) $\frac{3x^2-5}{9x^4-30x^2+26}$ (B) $\frac{3x^2-5}{9x^4-6x^2+26}$ (C) $\frac{3x^2}{x^4+2x^2-4}$ (D) $\frac{3x^2}{9x^4+30x^2-2}$

25. Which of the following functions from **Z** into **Z** are bijections?

(A) $f(x) = x^3$ (B) $f(x) = x + 2$ (C) $f(x) = 2x + 1$ (D) $f(x) = x^2 + 1$

26. Let f : **R** → **R** be the functions defined by $f(x) = x^3 + 5$. Then $f^{-1}(x)$ is

(A) $(x+5)^{\frac{1}{3}}$ (B) $(x-5)^{\frac{1}{3}}$ (C) $(5-x)^{\frac{1}{3}}$ (D) $5 - x$

27. Let f : A → B and g : B → C be the bijective functions. Then $(gof)^{-1}$ is

(A) $f^{-1}og^{-1}$ (B) fog (C) $g^{-1}of^{-1}$ (D) gof

28. Let f : **R** $\left\{\frac{3}{5}\right\}$ → **R** be defined by $f(x) = \frac{3x+2}{5x-3}$. Then

(A) $f^{-1}(x) = f(x)$ (B) $f^{-1}(x) = -f(x)$ (C) $(fof)\, x = -x$ (D) $f^{-1}(x) = \frac{1}{19} f(x)$

29. Let f : [0, 1] → [0, 1] be defined by $f(x) = \begin{cases} x & if\ x \text{ is rational} \\ 1-x & if\ x \text{ is irrational} \end{cases}$. Then $(fof)\, x$ is

(A) constant (B) $1 + x$ (C) x (D) none of these

30. Let f : [2, ∞) → **R** be the function defined by $f(x) = x^2 - 4x + 5$, then the range of f is

(A) **R** (B) [1, ∞) (C) [4, ∞) (D) [5, ∞)

31. Let f : **N** → **R** be the function defined by $f(x) = \frac{2x-1}{2}$ and g : **Q** → **R** be another function defined by $g(x) = x + 2$. Then $(gof)\ \frac{3}{2}$ is

(A) 1 (B) 1 (C) $\frac{7}{2}$ (D) none of these

32. Let f : **R** → **R** be defined by $f(x) = \begin{cases} 2x : x > 3 \\ x^2 : 1 < x \le 3 \\ 3x : x \le 1 \end{cases}$. Then $f(-1) + f(2) + f(4)$ is

(A) 9 (B) 14 (C) 5 (D) none of these

33. Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be given by $f(x) = \tan x$. Then f^{-1} (1) is

(A) $\frac{\pi}{4}$ (B) $\left\{n\pi + \frac{\pi}{4} : \text{n} \in \text{Z}\right\}$ (C) Does not exist (D) None of these

FILL IN THE BLANKS

34. Let f = {(1, 2), (3, 5), (4, 1) and g = {(2, 3), (5, 1), (1, 3)}. Then $g \circ f$ = ___________ and $f \circ g$ = ______.

35. Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be defined by $f(x)$ $\frac{x}{\sqrt{1+x^2}}$. Then ($f \circ f \circ f$) (x) = _______.

36. If $f(x) = [4 - (x - 7)^3\}$, then $f^{-1}(x)$ = _______.

TRUE OR FALSE

37. Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be the function defined by $f(x) = \sin(3x + 2)\ \forall\ x$ R. Then f is invertible.

38. Let A = {0, 1} and **N** be the set of natural numbers. Then the mapping $f : \mathbf{N} \rightarrow$ A defined by $f(2n - 1) = 0$, $f(2n) = 1$, $\forall\ n \in \mathbf{N}$, is onto.

39. The composition of functions is commutative.

40. The composition of functions is associative.

41. Every function is invertible.

ANSWERS

1. D = [-5, 5] **2.** $4x^2 + 4x - 1$ **3.** $f^{-1}(x) = \frac{x+3}{2}$

4. f^{-1} = {(b, a), (d, b), (a, c), (c, d)} **5.** $f\{f(x)\} = x^4 - 6x^3 + 10x^2 - 3x$ **6.** $\alpha = 2$, $\beta = -1$

7. (i) represents function which is surjective but not injective. (ii) does not represent function.

8. fog = {(2,5), (5,2), (1,5)}

11. (i) f is not function (ii) g is function (iii) h is function (iv) k is not function

14. $\frac{1}{3}, 1$

17. (i) f is one-one but not onto , (ii) g is neither one-one nor onto (iii) h is bijective, (iv) k is neither one-one nor onto.

19. (i) $4x^2 - 6x + 1$ (ii) $2x^2 + 6x - 1$ (iii) $x^4 + 6x^3 + 14x^2 + 15x + 5$ (iv) $4x - 9$ **20.** A

21. C **22.** B **23.** D **24.** A **25.** B **26.** B **27.** A **28.** A

29. C **30.** B **31.** D **32.** A **33.** B

34. gof = {(1, 3), (3, 1), (4, 3)} and fog = {(2, 5), (5, 2), (1, 5)} **35.** $\frac{x}{\sqrt{3x^2+1}}$

36. $f^{-1}(x) = 7 + (4 - x)^{1/3}$ **37.** False **38.** True **39.** False **40.** True **41.** False