4.3 EXERCISE

SHORT ANSWER TYPE QUESTIONS

Using the properties of determinants in Exercise 1 to 6, evaluate:

Q1. $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

Sol. Let $\Delta = \begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

$C_1 \to C_1 - C_2$

$$= \begin{vmatrix} x^2 - 2x + 2 & x - 1 \\ 0 & x + 1 \end{vmatrix}$$

$$= (x + 1)\,(x^2 - 2x + 2) - 0$$

$$= x^3 - 2x^2 + 2x + x^2 - 2x + 2 = x^3 - x^2 + 2$$

Q2. $\begin{vmatrix} a + x & y & z \\ x & a + y & z \\ x & y & a + z \end{vmatrix}$

Sol. Let $\Delta = \begin{vmatrix} a + x & y & z \\ x & a + y & z \\ x & y & a + z \end{vmatrix}$

$C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} a + x + y + z & y & z \\ a + x + y + z & a + y & z \\ a + x + y + z & y & a + z \end{vmatrix} = (a + x + y + z)\begin{vmatrix} 1 & y & z \\ 1 & a + y & z \\ 1 & y & a + z \end{vmatrix}$$

(Taking $a + x + y + z$ common)

$R_1 \to R_1 - R_2,\ R_2 \to R_2 - R_3$

$$= (a + x + y + z)\begin{vmatrix} 0 & -a & 0 \\ 0 & a & -a \\ 1 & y & a + z \end{vmatrix}$$

Expanding along $C_1 = (a + x + y + z)\left|1(a^2 - 0)\right| = a^2(a + x + y + z)$

Q3. $\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix}$

Sol. Let $\Delta = \begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix}$

Taking x^2, y^2 and z^2 common from C_1, C_2 and C_3 respectively

$$= x^2y^2z^2 \begin{vmatrix} 0 & x & x \\ y & 0 & y \\ z & z & 0 \end{vmatrix}$$

Expanding along R_1

$$= x^2y^2z^2 \left[0\begin{vmatrix} 0 & y \\ z & 0 \end{vmatrix} - x\begin{vmatrix} y & y \\ z & 0 \end{vmatrix} + x\begin{vmatrix} y & 0 \\ z & z \end{vmatrix} \right]$$

$$= x^2y^2z^2 \left[-x(0 - yz) + x(yz - 0) \right]$$

$$= x^2y^2z^2(xyz + xyz) = x^2y^2z^2(2xyz) = 2x^3y^3z^3$$

Q4. $\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$

Sol. Let $\Delta = \begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$

$C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} x+y+z & -x+y & -x+z \\ x+y+z & 3y & z-y \\ x+y+z & y-z & 3z \end{vmatrix}$$

Taking $(x + y + z)$ common from C_1

$$= (x+y+z)\begin{vmatrix} 1 & -x+y & -x+z \\ 1 & 3y & z-y \\ 1 & y-z & 3z \end{vmatrix}$$

$R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$

$$= (x+y+z)\begin{vmatrix} 0 & -x-2y & -x+y \\ 0 & 2y+z & -y-2z \\ 1 & y-z & 3z \end{vmatrix}$$

Expanding along C_1

$$= (x+y+z)\left[1\begin{vmatrix} -x-2y & -x+y \\ 2y+z & -y-2z \end{vmatrix}\right]$$

$$= (x+y+z)\left[(-x-2y)(-y-2z)-(2y+z)(-x+y)\right]$$

$$= (x+y+z)(xy+2zx+2y^2+4yz+2xy-2y^2+zx-zy)$$

$$= (x+y+z)(3xy+3zx+3yz) = 3(x+y+z)(xy+yz+zx)$$

Q5. $\begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$

Sol. Let $\Delta = \begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$

$C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 3x+4 & x & x \\ 3x+4 & x+4 & x \\ 3x+4 & x & x+4 \end{vmatrix}$$

Taking $(3x+4)$ common from C_1

$$= (3x+4)\begin{vmatrix} 1 & x & x \\ 1 & x+4 & x \\ 1 & x & x+4 \end{vmatrix}$$

$R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$

$$= (3x+4)\begin{vmatrix} 0 & -4 & 0 \\ 0 & 4 & -4 \\ 1 & x & x+4 \end{vmatrix}$$

Expanding along C_1

$$= (3x+4)\left[1\begin{vmatrix} -4 & 0 \\ 4 & -4 \end{vmatrix}\right] = (3x+4)(16-0) = 16(3x+4)$$

Q6. $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

Sol. Let $\Delta = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$R_1 \rightarrow R_1 + R_2 + R_3$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Taking $(a + b + c)$ common from R_1

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$C_1 \rightarrow C_1 - C_2$, $C_2 \rightarrow C_2 - C_3$

$$= (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ b+c+a & -(b+c+a) & 2b \\ 0 & a+b+c & c-a-b \end{vmatrix}$$

Taking $(b + c + a)$ from C_1 and C_2

$$= (a+b+c)^3\begin{vmatrix} 0 & 0 & 1 \\ 1 & -1 & 2b \\ 0 & 1 & c-a-b \end{vmatrix}$$

Expanding along R_1

$$= (a+b+c)^3\left[1\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}\right] = (a+b+c)^3.$$

Using the properties of determinants in Exercises 7 to 9, prove that:

Q7. $\begin{vmatrix} y^2z^2 & yz & y+z \\ z^2x^2 & zx & z+x \\ x^2y^2 & xy & x+y \end{vmatrix} = 0$

Sol. L.H.S. $= \begin{vmatrix} y^2z^2 & yz & y+z \\ z^2x^2 & zx & z+x \\ x^2y^2 & xy & x+y \end{vmatrix}$

$R_1 \rightarrow xR_1$, $R_2 \rightarrow yR_2$, $R_3 \rightarrow zR_3$ and dividing the determinant by xyz.

$$= \frac{1}{xyz}\begin{vmatrix} xy^2z^2 & xyz & xy+zx \\ yz^2x^2 & yzx & yz+xy \\ zx^2y^2 & zxy & zx+zy \end{vmatrix}$$

Taking xyz common from C_1 and C_2

$$= \frac{xyz \cdot xyz}{xyz}\begin{vmatrix} yz & 1 & xy+zx \\ zx & 1 & yz+xy \\ xy & 1 & zx+zy \end{vmatrix}$$

$C_3 \to C_3 + C_1$

$$= xyz\begin{vmatrix} yz & 1 & xy+yz+zx \\ zx & 1 & xy+yz+zx \\ xy & 1 & xy+yz+zx \end{vmatrix}$$

Taking $(xy + yz + zx)$ common from C_3

$$= (xyz)\,(xy+yz+zx)\begin{vmatrix} yz & 1 & 1 \\ zx & 1 & 1 \\ xy & 1 & 1 \end{vmatrix}$$

$$= (xyz)\,(xy+yz+zx)\begin{vmatrix} yz & 1 & 1 \\ zx & 1 & 1 \\ xy & 1 & 1 \end{vmatrix} = 0$$

[$\because$ C_2 and C_3 are identical]

L.H.S. = R.H.S. Hence proved.

Q8. $\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} = 4xyz$

Sol. L.H.S. $= \begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix}$

$C_1 \to C_1 - (C_2 + C_3)$

$$= \begin{vmatrix} 0 & z & y \\ -2x & z+x & x \\ -2x & x & x+y \end{vmatrix}$$

Taking –2 common from C_1

$$= -2\begin{vmatrix} 0 & z & y \\ x & z+x & x \\ x & x & x+y \end{vmatrix}$$

$R_2 \to R_2 - R_3$

$$= -2\begin{vmatrix} 0 & z & y \\ 0 & z & -y \\ x & x & x+y \end{vmatrix}$$

Expanding along C_1

$$= -2\left[x\left|-zy - zy\right|\right] = -2(-2xyz) = 4xyz \text{ R.H.S.}$$

L.H.S. = R.H.S.

Hence, proved.

Q9. $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$

Sol. L.H.S.= $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$

$R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$

$$= \begin{vmatrix} a^2-1 & a-1 & 0 \\ 2a-2 & a-1 & 0 \\ 3 & 3 & 1 \end{vmatrix} = \begin{vmatrix} (a+1)(a-1) & a-1 & 0 \\ 2(a-1) & a-1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

Taking $(a-1)$ common from C_1 and C_2

$$= (a-1)(a-1)\begin{vmatrix} a+1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

Expanding along C_3

$$= (a-1)^2\left[1\begin{vmatrix} a+1 & 1 \\ 2 & 1 \end{vmatrix}\right]$$

$$= (a-1)^2(a+1-2) = (a-1)^2(a-1) = (a-1)^3 \text{ R.H.S.}$$

L.H.S.= R.H.S.

Hence, proved.

Q10. If $A + B + C = 0$, then prove that

$$\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0$$

Sol. L.H.S. = $\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix}$

Expanding along C_1

$$= 1\begin{vmatrix} 1 & \cos A \\ \cos A & 1 \end{vmatrix} - \cos C\begin{vmatrix} \cos C & \cos B \\ \cos A & 1 \end{vmatrix} + \cos B\begin{vmatrix} \cos C & \cos B \\ 1 & \cos A \end{vmatrix}$$

$= 1(1 - \cos^2 A) - \cos C(\cos C - \cos A \cos B)$
$\quad + \cos B (\cos A \cos C - \cos B)$
$= \sin^2 A - \cos^2 C + \cos A \cos B \cos C$
$\quad + \cos A \cos B \cos C - \cos^2 B$
$= \sin^2 A - \cos^2 B - \cos^2 C + 2 \cos A \cos B \cos C$
$= - \cos (A + B) \cdot \cos (A - B) - \cos^2 C + 2 \cos A \cos B \cos C$
$[\because \quad \sin^2 A - \cos^2 B = - \cos (A + B) \cdot \cos (A - B)]$
$= - \cos (- C) \cdot \cos (A - B) + \cos C (2 \cos A \cos B - \cos C)$
$[\because \quad A + B + C = 0]$
$= - \cos C(\cos A \cos B + \sin A \sin B)$
$\quad + \cos C(2 \cos A \cos B - \cos C)$
$= - \cos C(\cos A \cos B + \sin A \sin B - 2 \cos A \cos B + \cos C)$
$= - \cos C(- \cos A \cos B + \sin A \sin B + \cos C)$
$= \cos C(\cos A \cos B - \sin A \sin B - \cos C)$
$= \cos C[\cos(A + B) - \cos C]$
$= \cos C[\cos (- C) - \cos C] \qquad [\because \quad A + B = - C]$
$= \cos C[\cos C - \cos C] = \cos C \cdot 0 = 0$ R.H.S.

L.H.S. = R.H.S.

Hence, proved.

Q11. If the coordinates of the vertices of an equilateral triangle with sides of length 'a' are (x_1, y_1), (x_2, y_2) and (x_3, y_3), then

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \frac{3a^4}{4}.$$

Sol. Area of triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3)

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Let $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \Rightarrow \Delta^2 = \frac{1}{4}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2$

But area of equilateral triangle whose side is 'a' = $\frac{\sqrt{3}}{4} a^2$

$$\therefore \qquad \left(\frac{\sqrt{3}}{4} a^2\right)^2 = \frac{1}{4}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2$$

$$\Rightarrow \frac{3}{16}a^4 = \frac{1}{4}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 \quad \Rightarrow \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \frac{3}{16}a^4 \times 4 = \frac{3}{4}a^4$$

Hence, proved.

Q12. Find the value of θ satisfying $\begin{bmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{bmatrix} = 0$.

Sol. Let $A = \begin{bmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{bmatrix} = 0$

$$|A| = \begin{vmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{vmatrix} = 0$$

$C_1 \to C_1 - C_2$

$$\Rightarrow \begin{vmatrix} 0 & 1 & \sin 3\theta \\ -7 & 3 & \cos 2\theta \\ 14 & -7 & -2 \end{vmatrix} = 0$$

Taking 7 common from C_1

$$\Rightarrow 7\begin{vmatrix} 0 & 1 & \sin 3\theta \\ -1 & 3 & \cos 2\theta \\ 2 & -7 & -2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & 1 & \sin 3\theta \\ -1 & 3 & \cos 2\theta \\ 2 & -7 & -2 \end{vmatrix} = 0$$

Expanding along C_1

$$\Rightarrow 1\begin{vmatrix} 1 & \sin 3\theta \\ -7 & -2 \end{vmatrix} + 2\begin{vmatrix} 1 & \sin 3\theta \\ 3 & \cos 2\theta \end{vmatrix} = 0$$

$$\Rightarrow -2 + 7\sin 3\theta + 2(\cos 2\theta - 3\sin 3\theta) = 0$$

$$\Rightarrow -2 + 7\sin 3\theta + 2\cos 2\theta - 6\sin 3\theta = 0$$

$$\Rightarrow -2 + 2\cos 2\theta + \sin 3\theta = 0$$

$$\Rightarrow -2 + 2(1 - 2\sin^2\theta) + 3\sin\theta - 4\sin^3\theta = 0$$

$$\Rightarrow -2 + 2 - 4\sin^2\theta + 3\sin\theta - 4\sin^3\theta = 0$$

$$\Rightarrow \quad -4\sin^3\theta - 4\sin^2\theta + 3\sin\theta = 0$$

$$\Rightarrow \quad -\sin\theta(4\sin^2\theta + 4\sin\theta - 3) = 0$$

$$\sin\theta = 0 \quad \text{or} \quad 4\sin^2\theta + 4\sin\theta - 3 = 0$$

$$\therefore \quad \theta = n\pi \quad \text{or} \quad 4\sin^2\theta + 6\sin\theta - 2\sin\theta - 3 = 0$$

when $n \in$ I

$$\Rightarrow \quad 2\sin\theta(2\sin\theta + 3) - 1(2\sin\theta + 3) = 0$$

$$\Rightarrow \quad (2\sin\theta + 3)(2\sin\theta - 1) = 0$$

$$\Rightarrow \quad 2\sin\theta + 3 = 0 \quad \text{or} \quad 2\sin\theta - 1 = 0$$

$$\sin\theta = \frac{-3}{2} \quad \text{or} \quad \sin\theta = \frac{1}{2}$$

$\sin\theta = \dfrac{-3}{2}$ is not possible as $-1 \le x \le 1$

$$\therefore \quad \sin\theta = \frac{1}{2} \quad \Rightarrow \quad \sin\theta = \sin\frac{\pi}{6} \quad \Rightarrow \quad \theta = n\pi + (-1)^n \cdot \frac{\pi}{6}$$

Hence, $\theta = n\pi \quad \text{or} \quad n\pi + (-1)^n \dfrac{\pi}{6}$

Q13. If $\begin{bmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{bmatrix} = 0,$ then find values of x.

Sol. Let

$$A = \begin{bmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{bmatrix} = 0$$

$$|A| = \begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$$

$R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \quad \begin{vmatrix} 12+x & 12+x & 12+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$$

Taking $(12 + x)$ common from R_1,

$$\Rightarrow \quad (12+x)\begin{vmatrix} 1 & 1 & 1 \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$$

$C_1 \to C_1 - C_2,\ C_2 \to C_2 - C_3$

$$(12+x)\begin{vmatrix} 0 & 0 & 1 \\ 2x & -2x & 4+x \\ 0 & 2x & 4-x \end{vmatrix} = 0$$

Expanding along R_1

$$\Rightarrow \qquad (12+x)\left[1\cdot\begin{vmatrix} 2x & -2x \\ 0 & 2x \end{vmatrix}\right] = 0$$

$$(12+x)(4x^2-0) = 0 \Rightarrow 12+x=0 \quad \text{or} \quad 4x^2 = 0$$

$\Rightarrow \quad x = -12 \quad \text{or} \quad x = 0$

Q14. If $a_1, a_2, a_3, \ldots, a_r$ are in G.P., then prove that the determinant

$$\begin{vmatrix} a_{r+1} & a_{r+5} & a_{r+9} \\ a_{r+7} & a_{r+11} & a_{r+15} \\ a_{r+11} & a_{r+17} & a_{r+21} \end{vmatrix}$$ is independent of r.

Sol. If $a_1, a_2, a_3, \ldots a_r$ be the terms of G.P., then

$$a_n = AR^{n-1}$$

(where A is the first term and R is the common ratio of the G.P.)

$\therefore \quad a_{r+1} = AR^{r+1-1} = AR^r;\ a_{r+5} = AR^{r+5-1} = AR^{r+4}$

$a_{r+9} = AR^{r+9-1} = AR^{r+8};\ a_{r+7} = AR^{r+7-1} = AR^{r+6}$

$a_{r+11} = AR^{r+11-1} = AR^{r+10};\ a_{r+15} = AR^{r+15-1} = AR^{r+14}$

$a_{r+17} = AR^{r+17-1} = AR^{r+16};\ a_{r+21} = AR^{r+21-1} = AR^{r+20}$

$\therefore$ The determinant becomes

$$\begin{vmatrix} AR^r & AR^{r+4} & AR^{r+8} \\ AR^{r+6} & AR^{r+10} & AR^{r+14} \\ AR^{r+10} & AR^{r+16} & AR^{r+20} \end{vmatrix}$$

Taking AR^r, AR^{r+6} and AR^{r+10} common from R_1, R_2 and R_3 respectively.

$$AR^r \cdot AR^{r+6} \cdot AR^{r+10}\begin{vmatrix} 1 & R^4 & R^8 \\ 1 & R^4 & R^8 \\ 1 & R^6 & R^{10} \end{vmatrix}$$

$$= AR^r \cdot AR^{r+6} \cdot AR^{r+10}\,|0|$$

[$\because$ R_1 and R_2 are identical rows]

$= 0$

Hence, the given determinant is independent of r.

Q15. Show that the points $(a+5, a-4)$, $(a-2, a+3)$ and (a, a) do not lie on a straight line for any value of a.

Sol. If the given points lie on a straight line, then the area of the triangle formed by joining the points pairwise is zero.

So, $\begin{vmatrix} a+5 & a-4 & 1 \\ a-2 & a+3 & 1 \\ a & a & 1 \end{vmatrix}$

$R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$

$$\Rightarrow \begin{vmatrix} 7 & -7 & 0 \\ -2 & 3 & 0 \\ a & a & 1 \end{vmatrix}$$

Expanding along C_3

$$1 \cdot \begin{vmatrix} 7 & -7 \\ -2 & 3 \end{vmatrix} = 21 - 14 = 7 \text{ units}$$

As $7 \neq 0$. Hence,, the three points do not lie on a straight line for any value of a.

Q16. Show that the ΔABC is an isosceles triangle if the determinant

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A+\cos A & \cos^2 B+\cos B & \cos^2 C+\cos C \end{vmatrix} = 0.$$

Sol. $\begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A+\cos A & \cos^2 B+\cos B & \cos^2 C+\cos C \end{vmatrix} = 0$

$C_1 \to C_1 - C_2, C_2 \to C_2 - C_3$

$$\Rightarrow \begin{vmatrix} 0 & 0 & 1 \\ \cos A-\cos B & \cos B-\cos C & 1+\cos C \\ \cos^2 A+\cos A - \cos^2 B-\cos B & \cos^2 B+\cos B - \cos^2 C-\cos C & \cos^2 C+\cos C \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & 0 & 1 \\ \cos A-\cos B & \cos B-\cos C & 1+\cos C \\ \cos^2 A-\cos^2 B + \cos A-\cos B & \cos^2 B-\cos^2 C + \cos B-\cos C & \cos^2 C+\cos C \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & 0 & 1 \\ \cos A-\cos B & \cos B-\cos C & 1+\cos C \\ (\cos A+\cos B)\times(\cos A-\cos B)+(\cos A-\cos B) & (\cos B+\cos C)\times(\cos B-\cos C)+(\cos B+\cos C) & \cos^2 C+\cos C \end{vmatrix} = 0$$

Taking (cos A – cos B) and (cos B – cos C) common from C_1 and C_2 respectively.

$$\Rightarrow \quad (\cos A - \cos B)(\cos B - \cos C)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 1+\cos C \\ \cos A + \cos B + 1 & \cos B + \cos C + 1 & \cos^2 C + \cos C \end{vmatrix} = 0$$

Expanding along R_1

$$\Rightarrow (\cos A - \cos B)(\cos B - \cos C)\left[1\begin{vmatrix} 1 & 1 \\ \cos A + \cos B + 1 & \cos B + \cos C + 1 \end{vmatrix}\right] = 0$$

$$\Rightarrow (\cos A - \cos B)(\cos B - \cos C)\left[\begin{matrix}(\cos B + \cos C + 1) - \\ (\cos A + \cos B + 1)\end{matrix}\right] = 0$$

$$\Rightarrow (\cos A - \cos B)(\cos B - \cos C)\left[\cos B + \cos C + 1 - \cos A - \cos B - 1\right] = 0$$

$\Rightarrow$ (cos A – cos B) (cos B – cos C) (cos C – cos A) = 0

cos A – cos B = 0 or cos B – cos C = 0

or cos C – cos A = 0

$\Rightarrow$ cos A = cos B or cos B = cos C or cos C = cos A

$\Rightarrow \angle A = \angle C$ or $\angle B = \angle C \quad \Rightarrow \angle A = \angle B$

Hence, ΔABC is an isosceles triangle.

Q17. Find A^{-1} if $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and show that $A^{-1} = \dfrac{A^2 - 3I}{2}$.

Sol. Here, $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$|A| = 0\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$= 0 - 1(0 - 1) + 1(1 - 0)$

$= 1 + 1 = 2 \neq 0$ (non-singular matrix.)

Now, co-factors,

$$a_{11} = +\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1, \quad a_{12} = -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1, \quad a_{13} = +\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

$$a_{21} = -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1, \quad a_{22} = +\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1, \quad a_{23} = -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1$$

$$a_{31} = +\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1, \quad a_{32} = -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1, \quad a_{33} = +\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$$\text{Adj (A)} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}' = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\therefore \qquad A^{-1} = \frac{1}{|A|}\text{Adj (A)} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Now, $A^2 = A \cdot A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0+1+1 & 0+0+1 & 0+1+0 \\ 0+0+1 & 1+0+1 & 1+0+0 \\ 0+1+0 & 1+0+0 & 1+1+0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Hence, $A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Now, we have to prove that $A^{-1} = \dfrac{A^2 - 3I}{2}$

$$\text{R.H.S.} = \frac{\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}{2}$$

$$= \frac{\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}}{2} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$= A^{-1} = \text{L.H.S.}$

Hence, proved.

LONG ANSWER TYPE QUESTIONS

Q18. If $A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$, find A^{-1}. Using A^{-1}, solve the system of linear equations $x - 2y = 10$, $2x - y - z = 8$, $-2y + z = 7$.

Sol. Given that

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$$

$$|A| = 1\begin{vmatrix} -1 & -2 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} -2 & -2 \\ 0 & 1 \end{vmatrix} + 0\begin{bmatrix} -2 & -1 \\ 0 & -1 \end{bmatrix}$$

$$= 1(-1-2) - 2(-2-0) + 0$$

$$= -3 + 4 = 1 \neq 0 \text{ (non-singular matrix.)}$$

Now co-factors,

$$a_{11} = +\begin{vmatrix} -1 & -2 \\ -1 & 1 \end{vmatrix} = -3,\ a_{12} = -\begin{vmatrix} -2 & -2 \\ 0 & 1 \end{vmatrix} = 2,\ a_{13} = +\begin{vmatrix} -2 & -1 \\ 0 & -1 \end{vmatrix} = 2$$

$$a_{21} = -\begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} = -2,\ a_{22} = +\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,\ a_{23} = -\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} = 1$$

$$a_{31} = +\begin{vmatrix} 2 & 0 \\ -1 & -2 \end{vmatrix} = -4,\ a_{32} = -\begin{vmatrix} 1 & 0 \\ -2 & -2 \end{vmatrix} = 2,\ a_{33} = +\begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix} = 3$$

$$\text{Adj (A)} = \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix}' = \begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$\therefore \qquad A^{-1} = \frac{1}{|A|}\text{Adj(A)} = \frac{1}{1}\begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow \qquad A^{-1} = \begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

Now, the system of linear equations is given by $x - 2y = 10$, $2x - y - z = 8$ and $-2y + z = 7$, which is in the form of CX = D.

$$\begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$$

where $C = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $D = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$

$\because \quad (A^T)^{-1} = (A^{-1})^T$

$\therefore \quad C^T = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix} = A$

$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = C^{-1}D \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -30+16+14 \\ -20+8+7 \\ -40+16+21 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ -3 \end{bmatrix}$

Hence, $x = 0$, $y = -5$ and $z = -3$

Q19. Using matrix method, solve the system of equation $3x + 2y - 2z = 3$, $x + 2y + 3z = 6$, $2x - y + z = 2$.

Sol. Given that

$$3x + 2y - 2z = 3$$
$$x + 2y + 3z = 6$$
$$2x - y + z = 2$$

Let $A = \begin{bmatrix} 3 & 2 & -2 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$

$$|A| = 3\begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$$
$$= 3(2+3) - 2(1-6) - 2(-1-4)$$
$$= 15 + 10 + 10 = 35 \neq 0 \text{ non-singular matrix}$$

Now, co-factors,

$a_{11} = +\begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} = 5$, $a_{12} = -\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 5$, $a_{13} = +\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -5$

$a_{21} = -\begin{vmatrix} 2 & -2 \\ -1 & 1 \end{vmatrix} = 0$, $a_{22} = +\begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = 7$, $a_{23} = -\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} = 7$

$a_{31} = +\begin{vmatrix} 2 & -2 \\ 2 & 3 \end{vmatrix} = 10$, $a_{32} = -\begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix} = -11$, $a_{33} = +\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} = 4$

$$\text{Adj}(A) = \begin{bmatrix} 5 & 5 & -5 \\ 0 & 7 & 7 \\ 10 & -11 & 4 \end{bmatrix}' = \begin{bmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ -5 & 7 & 4 \end{bmatrix}$$

$$\therefore \qquad A^{-1} = \frac{1}{|A|}\text{Adj}(A) = \frac{1}{35}\begin{bmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ -5 & 7 & 4 \end{bmatrix}$$

Now, $\quad X = A^{-1}B$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ -5 & 7 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 15+0+20 \\ 15+42-22 \\ -15+42+8 \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 35 \\ 35 \\ 35 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Hence, $x = 1$, $y = 1$ and $z = 1$.

Q20. If $A = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$, then find BA and use this to solve the system of equations $y + 2z = 7$, $x - y = 3$ and $2x + 3y + 4z = 17$.

Sol. We have, $A = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

$$BA = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2+4+0 & 2-2+0 & -4+4+0 \\ 4-12+8 & 4+6-4 & -8-12+20 \\ 0-4+4 & 0+2-2 & 0-4+10 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 6\,I$$

$$\therefore \qquad B^{-1} = \frac{1}{6}A = \frac{1}{6}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$$

The given equations can be re-write as,

$x - y = 3$, $2x + 3y + 4z = 17$ and $y + 2z = 7$

$$\therefore \quad \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 6 + 34 - 28 \\ -12 + 34 - 28 \\ 6 - 17 + 35 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 12 \\ -6 \\ 24 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

Hence, $x = 2$, $y = -1$ and $z = 4$

Q21. If $a + b + c \neq 0$ and $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} = 0$, then prove that $a = b = c$.

Sol. Given that: $a + b + c \neq 0$ and $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} = 0$

$C_1 \to C_1 + C_2 + C_3$

$$\Rightarrow \quad \begin{bmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{bmatrix} = 0$$

$$\Rightarrow \quad (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} = 0$$ (Taking $a + b + c$ common from C_1)

$$\Rightarrow \quad a + b + c \neq 0 \quad \therefore \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} = 0$$

$R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$

$$\Rightarrow \quad \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 1 & a & b \end{vmatrix} = 0$$

Expanding along C_1

$$\Rightarrow \quad 1\begin{vmatrix} b-c & c-a \\ c-a & a-b \end{vmatrix} = 0$$

$$\Rightarrow \quad (b-c)(a-b) - (c-a)^2 = 0$$
$$\Rightarrow \quad ab - b^2 - ac + bc - c^2 - a^2 + 2ac = 0$$
$$\Rightarrow \quad -a^2 - b^2 - c^2 + ab + bc + ac = 0$$
$$\Rightarrow \quad a^2 + b^2 + c^2 - ab - bc - ac = 0$$
$$\Rightarrow \quad 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ac = 0$$

(Multiplying both sides by 2)

$$\Rightarrow \quad (a^2 + b^2 - 2ab) + (b^2 + c^2 - 2bc) + (a^2 + c^2 - 2ac) = 0$$
$$\Rightarrow \quad (a-b)^2 + (b-c)^2 + (a-c)^2 = 0$$

It is only possible when $(a-b)^2 = (b-c)^2 = (a-c)^2 = 0$

$\therefore \quad a = b = c$ Hence, proved.

Q22. Prove that $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ac-b^2 \end{vmatrix}$ is divisible by $a+b+c$ and find the quotient.

Sol. Let $\Delta = \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ac-b^2 \end{vmatrix}$

$C_1 \to C_1 + C_2 + C_3$

$$\Rightarrow \quad \begin{vmatrix} ab+bc+ac-a^2-b^2-c^2 & ca-b^2 & ab-c^2 \\ ab+bc+ac-a^2-b^2-c^2 & ab-c^2 & bc-a^2 \\ ab+bc+ac-a^2-b^2-c^2 & bc-a^2 & ac-b^2 \end{vmatrix}$$

Taking $ab + bc + ac - a^2 - b^2 - c^2$ common from C_1

$$(ab+bc+ac-a^2-b^2-c^2)\begin{vmatrix} 1 & ca-b^2 & ab-c^2 \\ 1 & ab-c^2 & bc-a^2 \\ 1 & bc-a^2 & ac-b^2 \end{vmatrix}$$

$R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$

$$\Rightarrow \quad (ab+bc+ac-a^2-b^2-c^2)\begin{vmatrix} 0 & ca-b^2-ab+c^2 & ab-c^2-bc+a^2 \\ 0 & ab-c^2-bc+a^2 & bc-a^2-ac+b^2 \\ 1 & bc-a^2 & ac-b^2 \end{vmatrix}$$

$$\Rightarrow \quad (ab+bc+ac-a^2-b^2-c^2)\begin{vmatrix} 0 & a(c-b)+(c+b)(c-b) & b(a-c)+(a+c)(a-c) \\ 0 & b(a-c)+(a+c)(a-c) & c(b-a)+(b+a)(b-a) \\ 1 & bc-a^2 & ac-b^2 \end{vmatrix}$$

$$\Rightarrow \quad (ab+bc+ac-a^2-b^2-c^2)\begin{vmatrix} 0 & (c-b)(a+b+c) & (a-c)(a+b+c) \\ 0 & (a-c)(a+b+c) & (b-a)(a+b+c) \\ 1 & bc-a^2 & ac-b^2 \end{vmatrix}$$

$$\Rightarrow \quad (ab+bc+ac-a^2-b^2-c^2)(a+b+c)(a+b+c)\begin{vmatrix} 0 & c-b & a-c \\ 0 & a-c & b-a \\ 1 & bc-a^2 & ac-b^2 \end{vmatrix}$$

$$\Rightarrow \quad (a+b+c)^2(ab+bc+ac-a^2-b^2-c^2)\begin{vmatrix} 0 & c-b & a-c \\ 0 & a-c & b-a \\ 1 & bc-a^2 & ac-b^2 \end{vmatrix}$$

Expanding along C_1

$$\Rightarrow (a+b+c)^2(ab+bc+ac-a^2-b^2-c^2)\left[1\begin{vmatrix} c-b & a-c \\ a-c & b-a \end{vmatrix}\right]$$

$$\Rightarrow (a+b+c)^2(ab+bc+ac-a^2-b^2-c^2)[(c-b)(b-a)-(a-c)^2]$$

$$\Rightarrow (a+b+c)^2(ab+bc+ac-a^2-b^2-c^2)(bc-ca-b^2+ab-a^2-c^2+2ac)$$

$$\Rightarrow (a+b+c)^2(ab+bc+ac-a^2-b^2-c^2)(ab+bc+ca-a^2-b^2-c^2)$$

$$\Rightarrow (a+b+c)^2(ab+bc+ac-a^2-b^2-c^2)^2$$

$$\Rightarrow (a+b+c)(a+b+c)(a^2+b^2+c^2-ab-bc-ac)^2$$

Hence, the given determinant is divisible by a + b + c and the quotient is

$$(a+b+c)(a^2+b^2+c^2-ab-bc-ac)^2$$

$$\Rightarrow \quad (a+b+c)(a^2+b^2+c^2-ab-bc-ac)(a^2+b^2+c^2-ab-bc-ac)$$

$$\Rightarrow \quad (a^3+b^3+c^3-3abc)(a^2+b^2+c^2-ab-bc-ac)$$

$$\Rightarrow \quad -(a^3 + b^3 + c^3 - 3abc)(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ac)$$

$$\Rightarrow \quad \frac{1}{2}(a^3 + b^3 + c^3 - 3abc)[(a-b)^2 + (b-c)^2 + (a-c)^2]$$

Q23. If $x + y + z = 0$, prove that $\begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix} = xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$

Sol. L.H.S.

Let $\Delta = \begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix}$

Expanding along R_1

$$\Rightarrow \quad xa\begin{vmatrix} za & xb \\ xc & ya \end{vmatrix} - yb\begin{vmatrix} yc & xb \\ zb & ya \end{vmatrix} + zc\begin{vmatrix} yc & za \\ zb & xc \end{vmatrix}$$

$$\Rightarrow \quad xa(yza^2 - x^2bc) - yb(y^2ac - xzb^2) + zc(xyc^2 - z^2ab)$$

$$\Rightarrow \quad xyza^3 - x^3abc - y^3abc + xyzb^3 + xyzc^3 - z^3abc$$

$$\Rightarrow \quad xyz(a^3 + b^3 + c^3) - abc(x^3 + y^3 + z^3)$$

$$\Rightarrow \quad xyz(a^3 + b^3 + c^3) - abc(3xyz)$$

$[(\because \quad x + y + z = 0)\ (\therefore \quad x^3 + y^3 + z^3 = 3xyz)]$

$$\Rightarrow \quad xyz(a^3 + b^3 + c^3 - 3abc)$$

R.H.S. $\quad xyz\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$

$R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \quad xyz\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$$\Rightarrow \quad xyz(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ c & a & b \\ b & c & a \end{vmatrix}$$ (Taking $a + b + c$ common from R_1)

$C_1 \to C_1 - C_2,\ C_2 \to C_2 - C_3$

$$\Rightarrow \quad xyz(a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ c-a & a-b & b \\ b-c & c-a & a \end{vmatrix}$$

Expanding along R_1

$$\Rightarrow \quad xyz(a+b+c)\left[1\begin{vmatrix} c-a & a-b \\ b-c & c-a \end{vmatrix}\right]$$

$$\Rightarrow \quad xyz(a+b+c)\left[(c-a)^2 - (b-c)(a-b)\right]$$

$$\Rightarrow \quad xyz(a+b+c)(c^2 + a^2 - 2ca - ab + b^2 + ac - bc)$$

$$\Rightarrow \quad xyz(a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

$$\Rightarrow \quad xyz(a^3 + b^3 + c^3 - 3abc)$$

$$[a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)]$$

L.H.S. = R.H.S.

Hence, proved.

OBJECTIVE TYPE QUESTIONS (M.C.Q.)

Choose the correct answer from given four options in each of the Exercises from 24 to 37.

Q24. If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, then the value of x is

(*a*) 3 (*b*) ± 3 (*c*) ± 6 (*d*) 6

Sol. Given that

$$\Rightarrow \quad \begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$$

$$\Rightarrow \quad 2x^2 - 40 = 18 + 14 \Rightarrow 2x^2 = 32 + 40$$

$$\Rightarrow \quad 2x^2 = 72 \Rightarrow x^2 = 36$$

$$\therefore \quad x = \pm 6$$

Hence, the correct option is (*c*).

Q25. The value of determinant $\begin{vmatrix} a-b & b+c & a \\ b-a & c+a & b \\ c-a & a+b & c \end{vmatrix}$ is

(*a*) $a^3 + b^3 + c^3$ (*b*) $3bc$

(*c*) $a^3 + b^3 + c^3 - 3abc$ (*d*) None of these

Sol. Here, we have $\begin{vmatrix} a-b & b+c & a \\ b-a & c+a & b \\ c-a & a+b & c \end{vmatrix}$

$C_2 \rightarrow C_2 + C_3$

$$\Rightarrow \begin{vmatrix} a-b & a+b+c & a \\ b-a & a+b+c & b \\ c-a & a+b+c & c \end{vmatrix}$$

$$\Rightarrow (a+b+c)\begin{vmatrix} a-b & 1 & a \\ b-a & 1 & b \\ c-a & 1 & c \end{vmatrix}$$ (Taking $a+b+c$ common from C_2)

$R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$

$$\Rightarrow (a+b+c)\begin{vmatrix} 2(a-b) & 0 & a-b \\ b-c & 0 & b-c \\ c-a & 1 & c \end{vmatrix}$$

Taking $(a-b)$ and $(b-c)$ common from R_1 and R_2 respectively

$$\Rightarrow (a+b+c)(a-b)(b-c)\begin{vmatrix} 2 & 0 & 1 \\ 1 & 0 & 1 \\ c-a & 1 & c \end{vmatrix}$$

Expanding along C_2

$$\Rightarrow (a+b+c)(a-b)(b-c)\left[-1\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}\right]$$

$$\Rightarrow (a+b+c)(a-b)(b-c)(-1)$$

$$\Rightarrow (a+b+c)(a-b)(c-b)$$

Hence, the correct option is (*d*).

Q26. The area of a triangle with vertices $(-3, 0)$, $(3, 0)$ and $(0, k)$ is 9 sq units. Then, the value of k will be

(*a*) 9 (*b*) 3 (*c*) −9 (*d*) 6

Sol. Area of triangle with vertices $(x_1\ y_1)$, $(x_2\ y_2)$ and (x_3, y_3) will be:

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \Rightarrow \Delta = \frac{1}{2}\begin{vmatrix} -3 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & k & 1 \end{vmatrix}$$

$$\Rightarrow \quad = \frac{1}{2}\left[-3\begin{vmatrix} 0 & 1 \\ k & 1 \end{vmatrix} - 0\begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & 0 \\ 0 & k \end{vmatrix}\right]$$

$$\Rightarrow \quad = \frac{1}{2}\left[-3(-k) - 0 + 1(3k)\right]$$

$$\Rightarrow \quad = \frac{1}{2}(3k+3k) \quad \Rightarrow \quad \frac{1}{2}(6k) = 3k$$

$$3k = 9 \quad \Rightarrow \quad k = 3$$

Hence, the correct option is (*b*).

Q27. The determinant $\begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix}$ equals

(*a*) $abc\,(b-c)\,(c-a)\,(a-b)$ (*b*) $(b-c)\,(c-a)\,(a-b)$

(*c*) $(a+b+c)\,(b-c)\,(c-a)\,(a-b)$ (*d*) None of these

Sol. Let $\Delta = \begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix}$

$= \begin{vmatrix} b(b-a) & b-c & c(b-a) \\ a(b-a) & a-b & b(b-a) \\ c(b-a) & c-a & a(b-a) \end{vmatrix}$ (Taking $(b-a)$ common from C_1 and C_3)

$= (b-a)^2 \begin{vmatrix} b & b-c & c \\ a & a-b & b \\ c & c-a & a \end{vmatrix}$

$C_1 \to C_1 - C_3$

$= (a-b)^2 \begin{vmatrix} b-c & b-c & c \\ a-b & a-b & b \\ c-a & c-a & a \end{vmatrix}$ (C_1 and C_2 are identical columns.)

$= (a-b)^2 \cdot 0$

$= 0$

Hence, the correct option is (*d*).

Q28. The number of distinct real roots of $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ in the interval $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$ is

(*a*) 0 (*b*) 2 (*c*) 1 (*d*) 3

Sol. Given that

$$\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$$

$C_1 \to C_1 + C_2 + C_3$

$$\Rightarrow \begin{vmatrix} 2\cos x + \sin x & \cos x & \cos x \\ 2\cos x + \sin x & \sin x & \cos x \\ 2\cos x + \sin x & \cos x & \sin x \end{vmatrix} = 0$$

Taking $2\cos x + \sin x$ common from C_1

$$\Rightarrow \quad (2\cos x + \sin x)\begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix} = 0$$

$R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$

$$\Rightarrow (2\cos x + \sin x)\begin{vmatrix} 0 & \cos x - \sin x & 0 \\ 0 & \sin x - \cos x & \cos x - \sin x \\ 1 & \cos x & \sin x \end{vmatrix} = 0$$

$$\Rightarrow (2\cos x + \sin x)\left[1\begin{vmatrix} \cos x - \sin x & 0 \\ \sin x - \cos x & \cos x - \sin x \end{vmatrix}\right]$$

$$\Rightarrow \quad (2\cos x + \sin x)(\cos x - \sin x)^2 = 0$$

$2\cos x + \sin x = 0$

$2 + \tan x = 0$

$\therefore \quad \tan x = -2$

But $\quad -\frac{\pi}{4} \le x \le \frac{\pi}{4}$

So, x has no solution.

$(\cos x - \sin x)^2 = 0$

$\cos x - \sin x = 0$

$\Rightarrow \quad \tan x = 1$

$\Rightarrow \quad \tan x = \tan \frac{\pi}{4}$

$\therefore \quad x = \frac{\pi}{4} \in \left[\frac{-\pi}{4}, \frac{\pi}{4}\right]$

So, it will have only one real root.

Hence, the correct option is (*c*).

Q29. If A, B and C are angles of a triangle, then the determinant $\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$ is equal to

(*a*) 0 (*b*) -1 (*c*) 1 (*d*) None of these

Sol. Let $\Delta = \begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$

$C_1 \to aC_1 + bC_2 + cC_3$

$$\Rightarrow \quad \begin{vmatrix} -a + b\cos C + c\cos B & \cos C & \cos B \\ a\cos C - b + c\,\text{Cos}\,A & -1 & \cos A \\ a\cos B + b\cos A - C & \cos A & -1 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} -a+a & \cos C & \cos B \\ -b+b & -1 & \cos A \\ -c+c & \cos A & -1 \end{vmatrix} \quad \begin{bmatrix} \because \text{ From projection formula} \\ a = b\cos C + c\cos B \\ b = a\cos C + c\cos A \\ c = b\cos A + a\cos B \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & \cos C & \cos B \\ 0 & -1 & \cos A \\ 0 & \cos A & -1 \end{bmatrix} = 0$$

Hence, the correct option is (*a*).

Q30. Let $f(t) = \begin{bmatrix} \cos t & t & 1 \\ 2\sin t & t & 2t \\ \sin t & t & t \end{bmatrix}$, then $\lim\limits_{t \to 0} \dfrac{f(t)}{t^2}$ is equal to

(*a*) 0 (*b*) -1 (*c*) 2 (*d*) 3

Sol. We have $f(t) = \begin{bmatrix} \cos t & t & 1 \\ 2\sin t & t & 2t \\ \sin t & t & t \end{bmatrix}$

Expanding along R_1

$$= \cos t \begin{vmatrix} t & 2t \\ t & t \end{vmatrix} - t \begin{vmatrix} 2\sin t & 2t \\ \sin t & t \end{vmatrix} + 1 \begin{vmatrix} 2\sin t & t \\ \sin t & t \end{vmatrix}$$

$$= \cos t(t^2 - 2t^2) - t(2t\sin t - 2t\sin t) + (2t\sin t - t\sin t)$$

$$= -t^2\cos t + t\sin t$$

$$\therefore \quad \frac{f(t)}{t^2} = \frac{-t^2\cos t + t\sin t}{t^2}$$

$$\Rightarrow \quad \frac{f(t)}{t^2} = -\cos t + \frac{\sin t}{t}$$

$$\Rightarrow \quad \lim_{t \to 0} \frac{f(t)}{t^2} = \lim_{t \to 0}(-\cos t) + \lim_{t \to 0} \frac{\sin t}{t} = -1 + 1 = 0$$

Hence, the correct option is (*a*).

Q31. The maximum value of

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1+\cos\theta & 1 & 1 \end{vmatrix} \text{ is} \qquad (\theta \text{ is real number})$$

(*a*) $\dfrac{1}{2}$ (*b*) $\dfrac{\sqrt{3}}{2}$ (*c*) $\sqrt{2}$ (*d*) $\dfrac{2\sqrt{3}}{4}$

Sol. Given that: $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1+\cos\theta & 1 & 1 \end{vmatrix}$

$C_1 \to C_1 - C_2, C_2 \to C_2 - C_3$

$$= \begin{vmatrix} 0 & 0 & 1 \\ -\sin\theta & \sin\theta & 1 \\ \cos\theta & 0 & 1 \end{vmatrix}$$

Expanding along R_1

$$= 1\begin{vmatrix} -\sin\theta & \sin\theta \\ \cos\theta & 0 \end{vmatrix} = -\sin\theta\cos\theta$$

$$\Rightarrow \quad = -\frac{1}{2}\cdot 2\sin\theta\cos\theta = -\frac{1}{2}\sin 2\theta$$

but maximum value of $\sin 2\theta = 1 \Rightarrow \left|-\frac{1}{2}\cdot 1\right| = \frac{1}{2}$

Hence, the correct option is (*a*).

Q32. If $f(x) = \begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix}$, then

(*a*) $f(a) = 0$ (*b*) $f(b) = 0$ (*c*) $f(0) = 0$ (*d*) $f(1) = 0$

Sol. Given that: $f(x) = \begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix}$

$$f(a) = \begin{vmatrix} 0 & 0 & a-b \\ 2a & 0 & a-c \\ a+b & a+c & 0 \end{vmatrix}$$

Expanding along $R_1 = (a-b)\begin{vmatrix} 2a & 0 \\ a+b & a+c \end{vmatrix}$

$$= (a-b)[2a(a+c)] = (a-b)\cdot 2a\cdot(a+c) \neq 0$$

$$f(b) = \begin{vmatrix} 0 & b-a & 0 \\ b+a & 0 & b-c \\ 2b & b+c & 0 \end{vmatrix}$$

Expanding along R_1

$$-(b-a)\begin{vmatrix} b+a & b-c \\ 2b & 0 \end{vmatrix}$$

$$= -(b-a)[(-2b)(b-c)] = 2b(b-a)(b-c) \neq 0$$

$$f(0) = \begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

Expanding along R_1 = $a\begin{vmatrix} a & -c \\ b & 0 \end{vmatrix} - b\begin{vmatrix} a & 0 \\ b & c \end{vmatrix}$

$= a(bc) - b(ac) = abc - abc = 0$

Hence, the correct option is (*c*).

Q33. If $A = \begin{bmatrix} 2 & \lambda & -3 \\ 0 & 2 & 5 \\ 1 & 1 & 3 \end{bmatrix}$, then A^{-1} exists if

(*a*) $\lambda = 2$ (*b*) $\lambda \neq 2$ (*c*) $\lambda \neq -2$ (*d*) None of these

Sol. We have,

$$A = \begin{bmatrix} 2 & \lambda & -3 \\ 0 & 2 & 5 \\ 1 & 1 & 3 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 2 & \lambda & -3 \\ 0 & 2 & 5 \\ 1 & 1 & 3 \end{vmatrix}$$

Expanding along R_1 = $2\begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} - \lambda\begin{vmatrix} 0 & 5 \\ 1 & 3 \end{vmatrix} - 3\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix}$

$= 2(6-5) - \lambda(0-5) - 3(0-2)$

$= 2 + 5\lambda + 6 = 8 + 5\lambda$

If A^{-1} exists then $|A| \neq 0$

$\therefore$ $8 + 5\lambda \neq 0$ so $\lambda \neq \dfrac{-8}{5}$

Hence, the correct option is (*d*).

Q34. If A and B are invertible matrices, then which of the following is not correct?

(*a*) $\text{adj } A = |A| \cdot A^{-1}$ (*b*) $\det(A)^{-1} = [\det(A)]^{-1}$

(*c*) $(AB)^{-1} = B^{-1}A^{-1}$ (*d*) $(A+B)^{-1} = B^{-1} + A^{-1}$

Sol. If A and B are two invertible matrices then

(*a*) $\text{adj } A = |A| \cdot A^{-1}$ is correct

(*b*) $\det(A)^{-1} = [\det(A)]^{-1} = \dfrac{1}{\det(A)}$ is correct

(*c*) Also, $(AB)^{-1} = B^{-1}A^{-1}$ is correct

(*d*) $(A+B)^{-1} = \dfrac{1}{|A+B|} \cdot \text{adj}(A+B)$

$\therefore$ $(A+B)^{-1} \neq B^{-1} + A^{-1}$

Hence, the correct option is (*d*).

Q35. If x, y, z are all different from zero an $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = 0$, then the value of $x^{-1} + y^{-1} + z^{-1}$ is

(*a*) xyz (*b*) $x^{-1}y^{-1}z^{-1}$ (*c*) $-x-y-z$ (*d*) -1

Sol. Given that

$$\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = 0$$

Taking x, y and z common from R_1, R_2 and R_3 respectively.

$$\Rightarrow \quad xyz\begin{vmatrix} \frac{1}{x}+1 & \frac{1}{x} & \frac{1}{x} \\ \frac{1}{y} & \frac{1}{y}+1 & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z} & \frac{1}{z}+1 \end{vmatrix} = 0$$

$R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \quad xyz\begin{vmatrix} \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1 & \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1 & \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1 \\ \frac{1}{y} & \frac{1}{y}+1 & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z} & \frac{1}{z}+1 \end{vmatrix} = 0$$

Taking $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1$ common from R_1

$$\Rightarrow \quad xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{y} & \frac{1}{y}+1 & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z} & \frac{1}{z}+1 \end{vmatrix} = 0$$

$C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$$\Rightarrow \quad xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1\right)\begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & \frac{1}{y} \\ 0 & -1 & \frac{1}{z}+1 \end{vmatrix} = 0$$

Expanding along R_1

$$\Rightarrow \quad xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1\right)\left[1\begin{vmatrix}-1 & 1\\ 0 & -1\end{vmatrix}\right] = 0$$

$$\Rightarrow \quad xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1\right)(1) = 0$$

$$\Rightarrow \quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1=0 \text{ and } xyz \neq 0 \qquad (x \neq y \neq z \neq 0)$$

$$\therefore \quad x^{-1}+y^{-1}+z^{-1} = -1$$

Hence, the correct option is (*d*).

Q36. The value of the determinant $\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$ is

(*a*) $9x^2(x+y)$ (b) $9y^2(x+y)$ (c) $3y^2(x+y)$ (d) $7x^2(x+y)$

Sol. Let $\Delta = \begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$

$C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 3x+3y & x+y & x+2y \\ 3x+3y & x & x+y \\ 3x+3y & x+2y & x \end{vmatrix}$$

$$= (3x+3y)\begin{vmatrix} 1 & x+y & x+2y \\ 1 & x & x+y \\ 1 & x+2y & x \end{vmatrix}$$

[Taking $(3x+3y)$ common from C_1]

$R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$

$$\Rightarrow \quad 3(x+y)\begin{vmatrix} 0 & y & y \\ 0 & -2y & y \\ 1 & x+2y & x \end{vmatrix}$$

Expanding along C_1

$$\Rightarrow \quad 3(x+y)\left[1\begin{vmatrix} y & y \\ -2y & y \end{vmatrix}\right]$$

$\Rightarrow \quad 3(x+y)(y^2+2y^2) \Rightarrow 3(x+y)(3y^2) \Rightarrow 9y^2(x+y)$

Hence, the correct option is (*b*).

Q37. There are two values of '*a*' which makes determinant, $\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86$, then sum of these numbers is

(*a*) 4 (*b*) 5 (*c*) −4 (*d*) 9

Sol. Given that, $\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86$

Expanding along C_1

$$\Rightarrow 1\begin{vmatrix} a & -1 \\ 4 & 2a \end{vmatrix} - 2\begin{vmatrix} -2 & 5 \\ 4 & 2a \end{vmatrix} + 0\begin{vmatrix} -2 & 5 \\ a & -1 \end{vmatrix} = 86$$

$$\Rightarrow \quad (2a^2+4) - 2(-4a-20) = 86$$

$$\Rightarrow \quad 2a^2+4+8a+40 = 86$$

$$\Rightarrow \quad 2a^2+8a+4+40-86 = 0$$

$$\Rightarrow \quad 2a^2+8a-42 = 0$$

$$\Rightarrow \quad a^2+4a-21 = 0$$

$$\Rightarrow \quad a^2+7a-3a-21 = 0$$

$$\Rightarrow \quad a(a+7)-3(a+7) = 0$$

$$\Rightarrow \quad (a-3)(a+7) = 0$$

$\therefore \quad a = 3, -7$

Required sum of the two numbers = 3 − 7 = − 4.

Hence, the correct option is (*c*).

Fill in the Blanks

Q38. If A is a matrix of order 3 × 3, then |3A| = __________ .

Sol. We know that for a matrix of order 3 × 3,

$$|KA| = K^3 |A|$$

$$\therefore \quad |3A| = 3^3|A| = \mathbf{27}|\mathbf{A}|$$

Q39. If A is invertible matrix of order 3 × 3, then $|A^{-1}|$ __________ .

Sol. We know that for an invertible matrix A of any order, $|A^{-1}| = \dfrac{\mathbf{1}}{|\mathbf{A}|}$.

Q40. If $x, y, z \in$ R, then the value of determinant

$$\begin{vmatrix} (2^x + 2^{-x})^2 & (2^x - 2^{-x})^2 & 1 \\ (3^x + 3^{-x})^2 & (3^x - 3^{-x})^2 & 1 \\ (4^x + 4^{-x})^2 & (4^x - 4^{-x})^2 & 1 \end{vmatrix}$$ is equal to ________ .

Sol. We have, $$\begin{vmatrix} (2^x + 2^{-x})^2 & (2^x - 2^{-x})^2 & 1 \\ (3^x + 3^{-x})^2 & (3^x - 3^{-x})^2 & 1 \\ (4^x + 4^{-x})^2 & (4^x - 4^{-x})^2 & 1 \end{vmatrix}$$

$C_1 \to C_1 - C_2$

$$\Rightarrow \begin{vmatrix} (2^x + 2^{-x})^2 - (2^x - 2^{-x})^2 & (2^x - 2^{-x})^2 & 1 \\ (3^x + 3^{-x})^2 - (3^x - 3^{-x})^2 & (3^x - 3^{-x})^2 & 1 \\ (4^x + 4^{-x})^2 - (4^x - 4^{-x})^2 & (4^x - 4^{-x})^2 & 1 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} 4 \cdot 2^x \cdot 2^{-x} & (2^x - 2^{-x})^2 & 1 \\ 4 \cdot 3^x \cdot 3^{-x} & (3^x - 3^{-x})^2 & 1 \\ 4 \cdot 4^x \cdot 4^{-x} & (4^x - 4^{-x})^2 & 1 \end{vmatrix}$$ [applying $(a + b)^2 - (a - b)^2 = 4ab$]

$$\Rightarrow \begin{vmatrix} 4 & (2^x - 2^{-x})^2 & 1 \\ 4 & (3^x - 3^{-x})^2 & 1 \\ 4 & (4^x - 4^{-x})^2 & 1 \end{vmatrix}$$

$$\Rightarrow 4\begin{vmatrix} 1 & (2^x - 2^{-x})^2 & 1 \\ 1 & (3^x - 3^{-x})^2 & 1 \\ 1 & (4^x - 4^{-x})^2 & 1 \end{vmatrix}$$ (Taking 4 common from C_1)

$\Rightarrow$ $4 \cdot 0 = \mathbf{0}$ ($\because$ C_1 and C_3 are identical columns)

Q41. If $\cos 2\theta = 0$, then $$\begin{vmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix}^2 =$$ ________ .

Sol. Given that: $\cos 2\theta = 0$

$\Rightarrow$ $\cos 2\theta = \cos\frac{\pi}{2} \Rightarrow 2\theta = \frac{\pi}{2}$

$\therefore$ $\theta = \frac{\pi}{4}$

The determinant can be written as

$$\Rightarrow \begin{vmatrix} 0 & \cos\frac{\pi}{4} & \sin\frac{\pi}{4} \\ \cos\frac{\pi}{4} & \sin\frac{\pi}{4} & 0 \\ \sin\frac{\pi}{4} & 0 & \cos\frac{\pi}{4} \end{vmatrix}^2 \Rightarrow \begin{vmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{vmatrix}^2$$

$$\Rightarrow \left[\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}\right]^2 \quad \left(\text{Taking } \frac{1}{\sqrt{2}} \text{ common from } C_1, C_2 \text{ and } C_3\right)$$

Expanding along C_1,

$$\Rightarrow \left[\frac{1}{2\sqrt{2}}\left|-1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}+1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}\right|\right]^2 \Rightarrow \left[\frac{1}{2\sqrt{2}}\left|-1(1)+1(0-1)\right|\right]^2$$

$$\Rightarrow \left[\frac{1}{2\sqrt{2}}\left|-1-1\right|\right]^2 \Rightarrow \frac{1}{8}\cdot(4) = \mathbf{\frac{1}{2}}$$

Q42. If A is a matrix of order 3 × 3, then $(A^2)^{-1}$ = __________ .

Sol. For any square matrix A, $(A^2)^{-1}$ = $\mathbf{(A^{-1})^2}$.

Q43. If A is a matrix of order 3 × 3, then the number of minors in the determinants of A are __________ .

Sol. The order of a matrix is 3 × 3

$\therefore$ Total number of elements = 3 × 3 = 9

Hence, the number of minors in the determinant is **9**.

Q44. The sum of the products of elements of any row with the co-factors of corresponding elements is equal to __________ .

Sol. The sum of the products of elements of any row with the co-factors of corresponding elements is equal to **the value of the determinant of the given matrix**.

$$\text{Let } \Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding along R_1

$$a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\Rightarrow \quad a_{11}M_{11} + a_{12}M_{12} + a_{13}M_{13}$$

(where M_{11}, M_{12} and M_{13} are the minors of the corresponding elements)

Q45. If $x = -9$ is a root of $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$, then other two roots are ________ .

Sol. We have, $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$

Expanding along R_1

$$\Rightarrow \quad x\begin{vmatrix} x & 2 \\ 6 & x \end{vmatrix} - 3\begin{vmatrix} 2 & 2 \\ 7 & x \end{vmatrix} + 7\begin{vmatrix} 2 & x \\ 7 & 6 \end{vmatrix} = 0$$

$$\Rightarrow \quad x(x^2 - 12) - 3(2x - 14) + 7(12 - 7x) = 0$$

$$\Rightarrow \quad x^3 - 12x - 6x + 42 + 84 - 49x = 0$$

$$\Rightarrow \quad x^3 - 67x + 126 = 0 \qquad ...(1)$$

The roots of the equation may be the factors of 126 *i.e.*, 2 × 7 × 9

9 is given the root of the determinant put $x = 2$ in eq. (1)

$$(2)^3 - 67 \times 2 + 126 \quad \Rightarrow \quad 8 - 134 + 126 = 0$$

Hence, $\boldsymbol{x = 2}$ is the other root.

Now, put $x = 7$ in eq. (1)

$$(7)^3 - 67(7) + 126 \quad \Rightarrow \quad 343 - 469 + 126 = 0$$

Hence, $\boldsymbol{x = 7}$ is also the other root of the determinant.

Q46. $\begin{vmatrix} 0 & xyz & x-z \\ y-x & 0 & y-z \\ z-x & z-y & 0 \end{vmatrix} =$ ________ .

Sol. Let $\Delta = \begin{vmatrix} 0 & xyz & x-z \\ y-x & 0 & y-z \\ z-x & z-y & 0 \end{vmatrix}$

$C_1 \to C_1 - C_3$

$$= \begin{vmatrix} z-x & xyz & x-z \\ z-x & 0 & y-z \\ z-x & z-y & 0 \end{vmatrix}$$

Taking $(z - x)$ common from C_1

$$= (z-x)\begin{vmatrix} 1 & xyz & x-z \\ 1 & 0 & y-z \\ 1 & z-y & 0 \end{vmatrix}$$

$R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$

$$= (z-x)\begin{vmatrix} 0 & xyz & x-y \\ 0 & y-z & y-z \\ 1 & z-y & 0 \end{vmatrix}$$

Taking $(y - z)$ common from R_2

$$= (z-x)(y-z)\begin{vmatrix} 0 & xyz & x-y \\ 0 & 1 & 1 \\ 1 & z-y & 0 \end{vmatrix}$$

Expanding along C_1

$$= (z-x)(y-z)\left[1\begin{vmatrix} xyz & x-y \\ 1 & 1 \end{vmatrix}\right]$$

$$= (z-x)(y-z)(xyz - x + y) = \mathbf{(y-z)(z-x)(y-x+xyz)}$$

Q47. If $f(x) = \begin{vmatrix} (1+x)^{17} & (1+x)^{19} & (1+x)^{23} \\ (1+x)^{23} & (1+x)^{29} & (1+x)^{34} \\ (1+x)^{41} & (1+x)^{43} & (1+x)^{47} \end{vmatrix} = A + Bx + Cx^2 + \cdots$

then A = __________ .

Sol. Given that

$$\begin{vmatrix} (1+x)^{17} & (1+x)^{19} & (1+x)^{23} \\ (1+x)^{23} & (1+x)^{29} & (1+x)^{34} \\ (1+x)^{41} & (1+x)^{43} & (1+x)^{47} \end{vmatrix} = A + Bx + Cx^2 + \cdots$$

Taking $(1 + x)^{17}$, $(1 + x)^{23}$ and $(1 + x)^{41}$ common from R_1, R_2 and R_3 respectively

$$(1+x)^{17} \cdot (1+x)^{23} \cdot (1+x)^{41} \begin{vmatrix} 1 & (1+x)^{2} & (1+x)^{6} \\ 1 & (1+x)^{6} & (1+x)^{11} \\ 1 & (1+x)^{2} & (1+x)^{6} \end{vmatrix}$$

$\Rightarrow \quad (1+x)^{17} \cdot (1+x)^{23} \cdot (1+x)^{41} \cdot 0 \qquad$ (R_1 and R_3 are identical)

$\therefore \quad 0 = A + Bx + Cx^2 + \ldots$

By comparing the like terms, we get A = **0**.

State True or False for the statements of the following Exercises:

Q48. $(A^3)^{-1} = (A^{-1})^3$, where A is a square matrix and $|A| \neq 0$.

Sol. Since $(A^K)^{-1} = (A^{-1})^K$ where $K \in N$

So, $\qquad (A^3)^{-1} = (A^{-1})^3$ is true

Q49. $(a\text{A})^{-1} = \dfrac{1}{a}\text{A}^{-1}$, where a is any real number and A is a square matrix.

Sol. If A is a non-singular square matrix, then for any non-zero scalar 'a', aA is invertible.

$$\therefore \quad (a\text{A})\cdot\left(\frac{1}{a}\text{A}^{-1}\right) = a\cdot\frac{1}{a}\cdot\text{A}\cdot\text{A}^{-1} = \text{I}$$

So, $(a\text{A})$ is inverse of $\left(\dfrac{1}{a}\text{A}^{-1}\right)$

$\Rightarrow \quad (a\text{A})^{-1} = \dfrac{1}{a}\text{A}^{-1}$ is true.

Q50. $\left|\text{A}^{-1}\right| \neq |\text{A}|^{-1}$, where A is a non-singular matrix.

Sol. False.

Since $\left|\text{A}^{-1}\right| = |\text{A}|^{-1}$ for a non-singular matrix.

Q51. If A and B are matrices of order 3 and $|\text{A}| = 5, |\text{B}| = 3$ then $|3\text{AB}| = 27 \times 5 \times 3 = 405$

Sol. True.

$$|3\text{AB}| = 3^3|\text{AB}| = 27|\text{A}||\text{B}| = 27 \times 5 \times 3 \qquad [\because |\text{KA}| = \text{K}^n|\text{A}|]$$

Q52. If the value of a third order determinant is 12, then the value of the determinant formed by replacing each element by its co-factor will be 144.

Sol. True.

Since $|\text{A}| = 12$

If A is a square matrix of order n

then $\quad |\text{Adj A}| = |\text{A}|^{n-1}$

$\therefore \quad |\text{Adj A}| = |\text{A}|^{3-1} = |\text{A}|^2 = (12)^2 = 144 \qquad [n = 3]$

Q53. $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0$, where a, b, c are in A.P.

Sol. True.

Let $\quad \Delta = \begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$

$\text{R}_2 \to 2\text{R}_2 - (\text{R}_1 + \text{R}_3)$

$$= \begin{vmatrix} x+1 & x+2 & x+a \\ 0 & 0 & 2b-(a+c) \\ x+3 & x+4 & x+c \end{vmatrix}$$

a, b, c are in A.P.

$\therefore\ b - a = c - b \quad \Rightarrow \quad 2b = a + c$

$$= \begin{vmatrix} x+1 & x+2 & x+a \\ 0 & 0 & 0 \\ x+3 & x+4 & x+c \end{vmatrix} = 0$$

Q54. $|\text{adj A}| = |\text{A}|^2$, where A is a square matrix of order two.

Sol. False.

Since $|\text{adj A}| = |\text{A}|^{n-1}$ where n is the order of the square matrix.

Q55. The determinant $\begin{vmatrix} \sin A & \cos A & \sin A + \cos B \\ \sin B & \cos A & \sin B + \cos B \\ \sin C & \cos A & \sin C + \cos B \end{vmatrix}$ is equal to zero.

Sol. True.

Let $\Delta = \begin{vmatrix} \sin A & \cos A & \sin A + \cos B \\ \sin B & \cos A & \sin B + \cos B \\ \sin C & \cos A & \sin C + \cos B \end{vmatrix}$

Splitting up C_3

$$= \begin{vmatrix} \sin A & \cos A & \sin A \\ \sin B & \cos A & \sin B \\ \sin C & \cos A & \sin C \end{vmatrix} + \begin{vmatrix} \sin A & \cos A & \cos B \\ \sin B & \cos A & \cos B \\ \sin C & \cos A & \cos B \end{vmatrix}$$

$$= 0 + \begin{vmatrix} \sin A & \cos A & \cos B \\ \sin B & \cos A & \cos B \\ \sin C & \cos A & \cos B \end{vmatrix} \qquad [\because C_1 \text{ and } C_3 \text{ are identical}]$$

$$= \cos A \cos B \begin{vmatrix} \sin A & 1 & 1 \\ \sin B & 1 & 1 \\ \sin C & 1 & 1 \end{vmatrix}$$

[Taking cos A and cos B common from C_2 and C_3 respectively]

$= \cos A \cos B\ (0)$ $\qquad [\because C_2 \text{ and } C_3 \text{ are identical}]$

$= 0$

Q56. If the determinant $\begin{vmatrix} x+a & p+u & l+f \\ y+b & q+v & m+g \\ z+c & r+w & n+h \end{vmatrix}$ splits into exactly K determinants of order 3, each element of which contains only one term, then the value of K is 8.

Sol. True.

Let $\Delta = \begin{vmatrix} x+a & p+u & l+f \\ y+b & q+v & m+g \\ z+c & r+w & n+h \end{vmatrix}$

Splitting up C_1

$$\Rightarrow \begin{vmatrix} x & p+u & l+f \\ y & q+v & m+g \\ z & r+w & n+h \end{vmatrix} + \begin{vmatrix} a & p+u & l+f \\ b & q+v & m+g \\ c & r+w & n+h \end{vmatrix}$$

Splitting up C_2 in both determinants

$$\Rightarrow \begin{vmatrix} x & p & l+f \\ y & q & m+g \\ z & r & n+h \end{vmatrix} + \begin{vmatrix} x & u & l+f \\ y & v & m+g \\ z & w & n+h \end{vmatrix} + \begin{vmatrix} a & p & l+f \\ b & q & m+g \\ c & r & n+h \end{vmatrix}$$

$$+ \begin{vmatrix} a & u & l+f \\ b & v & m+g \\ c & w & n+h \end{vmatrix}$$

Similarly by splitting C_3 in each determinant, we will get 8 determinants.

Q57. Let $\Delta = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$ 16

then $\Delta_1 = \begin{vmatrix} p+x & a+x & a+p \\ q+y & b+y & b+q \\ r+z & c+z & c+r \end{vmatrix} = 32$

Sol. True.

Given that: $\Delta = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = 16$

L.H.S. $\Delta_1 = \begin{vmatrix} p+x & a+x & a+p \\ q+y & b+y & b+q \\ r+z & c+z & c+r \end{vmatrix}$

$C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 2p+2x+2a & a+x & a+p \\ 2q+2y+2b & b+y & b+q \\ 2r+2z+2c & c+z & c+r \end{vmatrix}$$

$$= 2\begin{vmatrix} p+x+a & a+x & a+p \\ q+y+b & b+y & b+q \\ r+z+c & c+z & c+r \end{vmatrix}$$ [Taking 2 common from C_1]

$$C_1 \to C_1 - C_2 = 2\begin{vmatrix} p & a+x & a+p \\ q & b+y & b+q \\ r & c+z & c+r \end{vmatrix}$$

$$C_3 \to C_3 - C_1{}^{d} = 2\begin{vmatrix} p & a+x & a \\ q & b+y & b \\ r & c+z & c \end{vmatrix}$$

Splitting up C_2

$$= 2\begin{vmatrix} p & a & a \\ q & b & b \\ r & c & c \end{vmatrix} + 2\begin{vmatrix} p & x & a \\ q & y & b \\ r & z & c \end{vmatrix} = 2\,(0) + 2\begin{vmatrix} p & x & a \\ q & y & b \\ r & z & c \end{vmatrix}$$

$$= 2\begin{vmatrix} p & x & a \\ q & y & b \\ r & z & c \end{vmatrix} \Rightarrow 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} \qquad (C_1 \leftrightarrow C_3 \text{ and } C_2 \leftrightarrow C_3)$$

$= 2 \times 16 = 32$

Q58. The maximum value of $\begin{vmatrix} 1 & 1 & 1 \\ 1 & (1+\sin\theta) & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$ is $\frac{1}{2}$.

Sol. True.

Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & (1+\sin\theta) & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$

$C_1 \to C_1 - C_2,\ C_2 \to C_2 - C_3$

$$= \begin{vmatrix} 0 & 0 & 1 \\ -\sin\theta & \sin\theta & 1 \\ 0 & -\cos\theta & 1+\cos\theta \end{vmatrix}$$

Expanding along C_3

$$= 1\begin{vmatrix} -\sin\theta & \sin\theta \\ 0 & -\cos\theta \end{vmatrix} = \sin\theta\cos\theta - 0 = \sin\theta\cos\theta$$

$$= \frac{1}{2}.2\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$$

$$= \frac{1}{2} \times 1 \qquad [\text{Maximum value of } \sin 2\theta = 1]$$

$$= \frac{1}{2}$$