10.3 EXERCISE

SHORT ANSWER TYPE QUESTIONS

Q1. Find the unit vector in the direction of sum of vectors $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{j} + \hat{k}$.

Sol. Given that

$$\vec{a} = 2\hat{i} - \hat{j} + \hat{k} \text{ and } \vec{b} = 2\hat{j} + \hat{k}$$

$$\vec{a} + \vec{b} = (2\hat{i} - \hat{j} + \hat{k}) + (2\hat{j} + \hat{k}) = 2\hat{i} + \hat{j} + 2\hat{k}$$

$\therefore$ Unit vector in the direction of $\vec{a} + \vec{b} = \dfrac{\vec{a} + \vec{b}}{|\vec{a} + \vec{b}|}$

$$= \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{(2)^2 + (1)^2 + (2)^2}} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{4 + 1 + 4}}$$

$$= \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{9}} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{3} = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}$$

Hence, the required unit vector is $\dfrac{2}{3}\hat{i} + \dfrac{1}{3}\hat{j} + \dfrac{2}{3}\hat{k}$.

Q2. If $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$, find the unit vector in the direction of (*i*) $6\vec{b}$ (*ii*) $2\vec{a} - \vec{b}$

Sol. Given that $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$

(*i*) $6\vec{b} = 6(2\hat{i} + \hat{j} - 2\hat{k}) = 12\hat{i} + 6\hat{j} - 12\hat{k}$

$\therefore$ Unit vector in the direction of $6\vec{b} = \dfrac{6\vec{b}}{|6\vec{b}|}$

$$= \frac{12\hat{i} + 6\hat{j} - 12\hat{k}}{\sqrt{(12)^2 + (6)^2 + (-12)^2}} = \frac{12\hat{i} + 6\hat{j} - 12\hat{k}}{\sqrt{144 + 36 + 144}}$$

$$= \frac{12\hat{i} + 6\hat{j} - 12\hat{k}}{\sqrt{324}} = \frac{12\hat{i} + 6\hat{j} - 12\hat{k}}{18}$$

$$= \frac{6}{18}(2\hat{i} + \hat{j} - 2\hat{k}) = \frac{1}{3}(2\hat{i} + \hat{j} - 2\hat{k})$$

Hence, the required unit vector is $\dfrac{1}{3}(2\hat{i} + \hat{j} - 2\hat{k})$.

(*ii*) $2\vec{a} - \vec{b} = 2(\hat{i} + \hat{j} + 2\hat{k}) - (2\hat{i} + \hat{j} - 2\hat{k})$

$$= 2\hat{i} + 2\hat{j} + 4\hat{k} - 2\hat{i} - \hat{j} + 2\hat{k} = \hat{j} + 6\hat{k}$$

$\therefore$ Unit vector in the direction of $2\vec{a} - \vec{b}$

$$= \frac{2\vec{a} - \vec{b}}{|2\vec{a} - \vec{b}|} = \frac{\hat{j} + 6\hat{k}}{\sqrt{(1)^2 + (6)^2}} = \frac{\hat{j} + 6\hat{k}}{\sqrt{1 + 36}}$$

$$= \frac{\hat{j} + 6\hat{k}}{\sqrt{37}} = \frac{1}{\sqrt{37}}[\hat{j} + 6\hat{k}]$$

Hence, the required unit vector is $\frac{1}{\sqrt{37}}[\hat{j} + 6\hat{k}]$.

Q3. Find a unit vector in the direction of $\overrightarrow{PQ}$, where P and Q have coordinates (5, 0, 8) and (3, 3, 2) respectively.

Sol. Given coordinates are P(5, 0, 8) and Q(3, 3, 2)

$\therefore$ $\overrightarrow{PQ} = (3-5)\hat{i} + (3-0)\hat{j} + (2-8)\hat{k} = -2\hat{i} + 3\hat{j} - 6\hat{k}$

$\therefore$ Unit vector in the direction of $\overrightarrow{PQ} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|}$

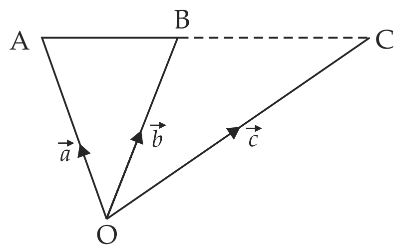
$$= \frac{-2\hat{i} + 3\hat{j} - 6\hat{k}}{\sqrt{(-2)^2 + (3)^2 + (-6)^2}} = \frac{-2\hat{i} + 3\hat{j} - 6\hat{k}}{\sqrt{4 + 9 + 36}} = \frac{-2\hat{i} + 3\hat{j} - 6\hat{k}}{\sqrt{49}}$$

$$= \frac{-2\hat{i} + 3\hat{j} - 6\hat{k}}{7} = \frac{1}{7}(-2\hat{i} + 3\hat{j} - 6\hat{k})$$

Hence, the required unit vector is $\frac{1}{7}(-2\hat{i} + 3\hat{j} - 6\hat{k})$.

Q4. If $\vec{a}$ and $\vec{b}$ are the position vectors of A and B respectively, find the position vector of a point C in BA produced such that BC = 1.5 BA.

Sol. Given that



$$BC = 1.5\,BA$$

$$\Rightarrow \quad \frac{BC}{BA} = 1.5 = \frac{3}{2}$$

$$\Rightarrow \quad \frac{\vec{c} - \vec{b}}{\vec{a} - \vec{b}} = \frac{3}{2}$$

$$\Rightarrow 2\vec{c} - 2\vec{b} = 3\vec{a} - 3\vec{b} \Rightarrow 2\vec{c} = 3\vec{a} - 3\vec{b} + 2\vec{b} \Rightarrow 2\vec{c} = 3\vec{a} - \vec{b}$$

$$\therefore \quad \vec{c} = \frac{3\vec{a} - \vec{b}}{2}$$

Hence, the required vector is $\vec{c} = \frac{3\vec{a} - \vec{b}}{2}$.

Q5. Using vectors, find the value of k, such that the points (k, –10, 3), (1, –1, 3) and (3, 5, 3) are collinear.

Sol. Let the given points are $A(k, -10, 3)$, $B(1, -1, 3)$ and $C(3, 5, 3)$

$$\overrightarrow{AB} = (1-k)\hat{i} + (-1+10)\hat{j} + (3-3)\hat{k}$$

$$\overrightarrow{AB} = (1-k)\hat{i} + 9\hat{j} + 0\hat{k}$$

$$\therefore \quad |\overrightarrow{AB}| = \sqrt{(1-k)^2 + (9)^2} = \sqrt{(1-k)^2 + 81}$$

$$\overrightarrow{BC} = (3-1)\hat{i} + (5+1)\hat{j} + (3-3)\hat{k} = 2\hat{i} + 6\hat{j} + 0\hat{k}$$

$$\therefore \quad |\overrightarrow{BC}| = \sqrt{(2)^2 + (6)^2} = \sqrt{4+36} = \sqrt{40} = 2\sqrt{10}$$

$$\overrightarrow{AC} = (3-k)\hat{i} + (5+10)\hat{j} + (3-3)\hat{k} = (3-k)\hat{i} + 15\hat{j} + 0\hat{k}$$

$$\therefore \quad |\overrightarrow{AC}| = \sqrt{(3-k)^2 + (15)^2} = \sqrt{(3-k)^2 + 225}$$

If A, B and C are collinear, then

$$|\overrightarrow{AB}| + |\overrightarrow{BC}| = |\overrightarrow{AC}|$$

$$\sqrt{(1-k)^2 + 81} + \sqrt{40} = \sqrt{(3-k)^2 + 225}$$

Squaring both sides, we have

$$\left[\sqrt{(1-k)^2 + 81} + \sqrt{40}\right]^2 = \left[\sqrt{(3-k)^2 + 225}\right]^2$$

$$\Rightarrow \quad (1-k)^2 + 81 + 40 + 2\sqrt{40}\sqrt{(1-k)^2 + 81} = (3-k)^2 + 225$$

$$\Rightarrow \quad 1 + k^2 - 2k + 121 + 2\sqrt{40}\sqrt{1 + k^2 - 2k + 81}$$
$$= 9 + k^2 - 6k + 225$$

$$\Rightarrow \quad 122 - 2k + 2\sqrt{40}\sqrt{k^2 - 2k + 82} = 234 - 6k$$

Dividing by 2, we get

$$\Rightarrow \quad 61 - k + \sqrt{40}\sqrt{k^2 - 2k + 82} = 117 - 3k$$

$$\Rightarrow \quad \sqrt{40}\sqrt{k^2 - 2k + 82} = 117 - 61 - 3k + k$$

$$\Rightarrow \sqrt{40}\sqrt{k^2 - 2k + 82} = 56 - 2k \Rightarrow 2\sqrt{10}\sqrt{k^2 - 2k + 82} = 56 - 2k$$

$$\Rightarrow \quad \sqrt{10}\sqrt{k^2 - 2k + 82} = 28 - k \qquad \text{(Dividing by 2)}$$

Squaring both sides, we get

$$\Rightarrow \quad 10(k^2 - 2k + 82) = 784 + k^2 - 56k$$

$$\Rightarrow \quad 10k^2 - 20k + 820 = 784 + k^2 - 56k$$

$$\Rightarrow \quad 10k^2 - k^2 - 20k + 56k + 820 - 784 = 0$$

$$\Rightarrow \quad 9k^2 + 36k + 36 = 0 \quad \Rightarrow \quad k^2 + 4k + 4 = 0 \quad \Rightarrow \quad (k+2)^2 = 0$$

$\Rightarrow \qquad k + 2 = 0 \quad \Rightarrow k = -2$

Hence, the required value is $k = -2$

Q6. A vector $\vec{r}$ is inclined at equal angles to the three axes. If the magnitude of $\vec{r}$ is $2\sqrt{3}$ units, find $\vec{r}$.

Sol. Since, the vector $\vec{r}$ makes equal angles with the axes, their direction cosines should be same

$\therefore \qquad l = m = n$

We know that $\quad l^2 + m^2 + n^2 = 1 \quad \Rightarrow l^2 + l^2 + l^2 = 1$

$$\Rightarrow \quad 3l^2 = 1 \quad \Rightarrow l^2 = \frac{1}{3} \quad \Rightarrow \quad l = \pm\frac{1}{\sqrt{3}}$$

$$\therefore \quad \hat{r} = \pm\frac{1}{\sqrt{3}}\hat{i} \pm \frac{1}{\sqrt{3}}\hat{j} \pm \frac{1}{\sqrt{3}}\hat{k} \quad \Rightarrow \hat{r} = \pm\frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$$

We know that $\quad \vec{r} = (\hat{r})|\vec{r}|$

$$= \pm\frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})\, 2\sqrt{3} = \pm 2(\hat{i} + \hat{j} + \hat{k})$$

Hence, the required value of $\vec{r}$ is $\pm 2(\hat{i} + \hat{j} + \hat{k})$.

Q7. A vector $\vec{r}$ has magnitude 14 and direction ratios 2, 3 and -6. Find the direction cosines and components of $\vec{r}$, given that $\vec{r}$ makes an acute angle with x-axis.

Sol. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors such that $\vec{a} = 2k, \vec{b} = 3k$ and $\vec{c} = -6k$

If l, m and n are the direction cosines of vector $\vec{r}$, then

$$l = \frac{\vec{a}}{|\vec{r}|} = \frac{2k}{14} = \frac{k}{7}$$

$$m = \frac{\vec{b}}{|\vec{r}|} = \frac{3k}{14} \quad \text{and} \quad n = \frac{\vec{c}}{|\vec{r}|} = \frac{-6k}{14} = \frac{-3k}{7}$$

We know that $l^2 + m^2 + n^2 = 1$

$$\therefore \quad \frac{k^2}{49} + \frac{9k^2}{196} + \frac{9k^2}{49} = 1$$

$$\Rightarrow \quad \frac{4k^2 + 9k^2 + 36k^2}{196} = 1 \quad \Rightarrow 49k^2 = 196 \quad \Rightarrow k^2 = 4$$

$$\therefore \quad k = \pm 2 \quad \text{and} \quad l = \frac{k}{7} = \frac{2}{7}$$

$$m = \frac{3k}{14} = \frac{3 \times 2}{14} = \frac{3}{7} \quad \text{and} \quad n = \frac{-3k}{7}\frac{-3 \times 2}{7} = \frac{-6}{7}$$

$$\therefore \quad \hat{r} = \pm\left(\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} - \frac{6}{7}\hat{k}\right)$$

$$\hat{r} = \hat{r}|\vec{r}|$$

$$\Rightarrow \quad \vec{r} = \pm\left(\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} - \frac{6}{7}\hat{k}\right)\cdot 14 = \pm(4\hat{i} + 6\hat{j} - 12\hat{k})$$

Hence, the required direction cosines are $\frac{2}{7}, \frac{3}{7}, \frac{-6}{7}$ and the components of $\vec{r}$ are $4\hat{i}, 6\hat{j}$ and $-12\hat{k}$.

Q8. Find a vector of magnitude 6, which is perpendicular to both the vectors $2\hat{i} - \hat{j} + 2\hat{k}$ and $4\hat{i} - \hat{j} + 3\hat{k}$.

Sol. Let $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = 4\hat{i} - \hat{j} + 3\hat{k}$

We know that unit vector perpendicular to $\vec{a}$ and $\vec{b} = \frac{(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 2 \\ 4 & -1 & 3 \end{vmatrix}$$

$$= \hat{i}(-3+2) - \hat{j}(6-8) + \hat{k}(-2+4) = -\hat{i} + 2\hat{j} + 2\hat{k}$$

$$\therefore \quad |\vec{a}\times\vec{b}| = \sqrt{(-1)^2 + (2)^2 + (2)^2} = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$\text{so,} \quad \frac{(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|} = \frac{-\hat{i} + 2\hat{j} + 2\hat{k}}{3} = \frac{1}{3}(-\hat{i} + 2\hat{j} + 2\hat{k})$$

Now the vector of magnitude $6 = \frac{1}{3}(-\hat{i} + 2\hat{j} + 2\hat{k})\cdot 6$

$$= 2(-\hat{i} + 2\hat{j} + 2\hat{k}) = -2\hat{i} + 4\hat{j} + 4\hat{k}$$

Hence, the required vector is $-2\hat{i} + 4\hat{j} + 4\hat{k}$.

Q9. Find the angle between the vectors $2\hat{i} - \hat{j} + \hat{k}$ and $3\hat{i} + 4\hat{j} - \hat{k}$.

Sol. Let $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 3\hat{i} + 4\hat{j} - \hat{k}$

and let θ be the angle between $\vec{a}$ and $\vec{b}$.

$$\therefore \quad \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{(2\hat{i} - \hat{j} + \hat{k})\cdot(3\hat{i} + 4\hat{j} - \hat{k})}{\sqrt{4+1+1}\cdot\sqrt{9+16+1}}$$

$$= \frac{6-4-1}{\sqrt{6}\cdot\sqrt{26}} \quad \Rightarrow \quad \frac{1}{2\sqrt{3}\cdot\sqrt{13}} = \frac{1}{2\sqrt{39}}$$

$$\therefore \quad \theta = \cos^{-1}\frac{1}{2\sqrt{39}} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{156}\right)$$

Hence, the required value of θ is $\cos^{-1}\left(\frac{1}{156}\right)$.

Q10. If $\vec{a} + \vec{b} + \vec{c} = 0$ show that $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$. Interpret the result geometrically.

Sol. Given that $\vec{a}+\vec{b}+\vec{c}=0$

So, $$\vec{a}\times(\vec{a}+\vec{b}+\vec{c}) = \vec{a}\times 0$$

$$\Rightarrow \quad \vec{a}\times\vec{a}+\vec{a}\times\vec{b}+\vec{a}\times\vec{c} = 0$$

$$\Rightarrow \quad \vec{o}+\vec{a}\times\vec{b}+\vec{a}\times\vec{c} = 0 \qquad (\vec{a}\times\vec{a}=0)$$

$$\Rightarrow \quad \vec{a}\times\vec{b}-\vec{c}\times\vec{a} = 0 \qquad (\vec{a}\times\vec{c}=-\vec{c}\times\vec{a})$$

$$\Rightarrow \quad \vec{a}\times\vec{b} = \vec{c}\times\vec{a} \qquad ...(i)$$

Now $$\vec{a}+\vec{b}+\vec{c} = 0$$

$$\Rightarrow \quad \vec{b}\times(\vec{a}+\vec{b}+\vec{c}) = \vec{b}\times 0$$

$$\Rightarrow \quad \vec{b}\times\vec{a}+\vec{b}\times\vec{b}+\vec{b}\times\vec{c} = 0$$

$$\Rightarrow \quad \vec{b}\times\vec{a}+\vec{o}+\vec{b}\times\vec{c} = 0 \qquad (\because \quad \vec{b}\times\vec{b}=0)$$

$$\Rightarrow \quad -(\vec{a}\times b)+\vec{b}\times\vec{c} = 0$$

$$\therefore \quad \vec{b}\times\vec{c} = \vec{a}\times\vec{b} \qquad ...(ii)$$

From eq. (*i*) and (*ii*) we get

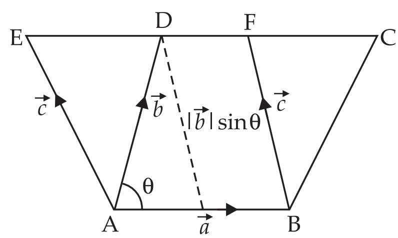
$\vec{a}\times\vec{b} = \vec{b}\times\vec{c}=\vec{c}\times\vec{a}$. Hence proved.

Geometrical Interpretation

According to figure, we have

Area of parallelogram ABCD is

$\Rightarrow \quad \left|\vec{a}\times\vec{b}\right| = |\vec{a}|\left|\vec{b}\right|\sin\theta$



Since, the parallelograms on the same base and between the same parallel lines are equal in area

$\therefore \left|\vec{a}\times\vec{b}\right|=\left|\vec{b}\times\vec{c}\right|=\left|\vec{c}\times\vec{a}\right|$

$\Rightarrow \vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$.

Q11. Find the sine of the angle between the vectors $\vec{a}=3\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}-2\hat{j}+4\hat{k}$.

Sol. Given that $\vec{a}=3\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}-2\hat{j}+4\hat{k}$

We know that $\left|\vec{a}\times\vec{b}\right| = |\vec{a}|\left|\vec{b}\right|\sin\theta$

$$\therefore \quad \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 2 & -2 & 4 \end{vmatrix}$$

$$= \hat{i}(4+4)-\hat{j}(12-4)+\hat{k}(-6-2)$$

$$= 8\hat{i}-8\hat{j}-8\hat{k}$$

$$\left|\vec{a}\times\vec{b}\right| = \sqrt{(8)^2+(-8)^2+(-8)^2}$$

$$= \sqrt{64+64+64} = \sqrt{192} = \sqrt{64 \times 3} = 8\sqrt{3}$$

$$|\vec{a}| = \sqrt{(3)^2 + (1)^2 + (2)^2} = \sqrt{9+1+4} = \sqrt{14}$$

$$|\vec{b}| = \sqrt{(2)^2 + (-2)^2 + (4)^2} = \sqrt{4+4+16}$$

$$= \sqrt{24} = 2\sqrt{6}$$

$$\therefore \qquad \sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} = \frac{8\sqrt{3}}{\sqrt{14} \cdot 2\sqrt{6}}$$

$$\Rightarrow \qquad = \frac{4\sqrt{3}}{\sqrt{84}} = \frac{4\sqrt{3}}{2\sqrt{21}} = \frac{2}{\sqrt{7}}$$

Hence, $\sin\theta = \dfrac{2}{\sqrt{7}}$.

Q12. If A, B, C, D are the points with position vectors $\hat{i} + \hat{j} - \hat{k}$, $2\hat{i} - \hat{j} + 3\hat{k}$, $2\hat{i} - 3\hat{k}$, $3\hat{i} - 2\hat{j} + \hat{k}$, respectively, find the projection of $\overline{AB}$ along $\overline{CD}$.

Sol. Here, Position vector of A = $\hat{i} + \hat{j} - \hat{k}$

Position vector of B = $2\hat{i} - \hat{j} + 3\hat{k}$

Position vector of C = $2\hat{i} - 3\hat{k}$

Position vector of D = $3\hat{i} - 2\hat{j} + \hat{k}$

$$\overrightarrow{AB} = \text{P.V of B} - \text{P.V of A}$$

$$= (2\hat{i} - \hat{j} + 3\hat{k}) - (\hat{i} + \hat{j} - \hat{k}) = \hat{i} - 2\hat{j} + 4\hat{k}$$

$$\overrightarrow{CD} = \text{P.V. of D} - \text{P.V. of C}$$

$$= (3\hat{i} - 2\hat{j} + \hat{k}) - (2\hat{i} - 3\hat{k}) = \hat{i} - 2\hat{j} + 4\hat{k}$$

$$\text{Projection of } \overrightarrow{AB} \text{ on } \overrightarrow{CD} = \frac{\overrightarrow{AB} \cdot \overrightarrow{CD}}{|\overrightarrow{CD}|}$$

$$= \frac{(\hat{i} - 2\hat{j} + 4\hat{k}) \cdot (\hat{i} - 2\hat{j} + 4\hat{k})}{\sqrt{(1)^2 + (-2)^2 + (4)^2}}$$

$$= \frac{1+4+16}{\sqrt{1+4+16}} = \frac{21}{\sqrt{21}} = \sqrt{21}$$

Hence, the required projection = $\sqrt{21}$.

Q13. Using vectors, find the area of the triangle ABC with vertices A(1, 2, 3), B(2, –1, 4) and C(4, 5, –1).

Sol. Given that A(1, 2, 3), B(2, –1, 4) and C(4, 5, –1)

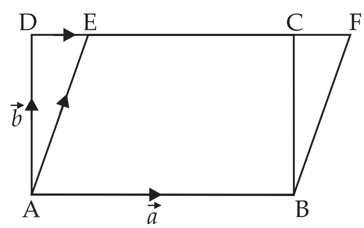
$$\overrightarrow{AB} = (2-1)\hat{i} + (-1-2)\hat{j} + (4-3)\hat{k}$$
$$\overrightarrow{AB} = \hat{i} - 3\hat{j} + \hat{k}$$
$$\overrightarrow{AC} = (4-1)\hat{i} + (5-2)\hat{j} + (-1-3)\hat{k} = 3\hat{i} + 3\hat{j} - 4\hat{k}$$

$$\text{Area of } \Delta ABC = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 3 & 3 & -4 \end{vmatrix}$$
$$= \frac{1}{2}[\hat{i}(12-3) - \hat{j}(-4-3) + \hat{k}(3+9)]$$
$$= \frac{1}{2}\left|9\hat{i} + 7\hat{j} + 12\hat{k}\right| = \frac{1}{2}\sqrt{(9)^2 + (7)^2 + (12)^2}$$
$$= \frac{1}{2}\sqrt{81 + 49 + 144} = \frac{1}{2}\sqrt{274}$$

Hence, the required area is $\frac{\sqrt{274}}{2}$.

Q14. Using vectors, prove that the parallelogram on the same base and between the same parallels, are equal in area.

Sol. Let ABCD and ABFE be two parallelograms on the same base AB and between same parallel lines AB and DF.



Let $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AD} = \vec{b}$

$\therefore$ Area of parallelogram ABCD $= \left|\vec{a} \times \vec{b}\right|$

Now Area of parallelogram ABFE $= \left|\overrightarrow{AB} \times \overrightarrow{AE}\right|$

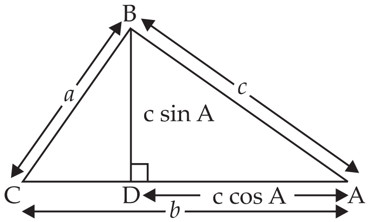
$$= \left|\vec{a} \times (\overrightarrow{AD} + \overrightarrow{DE})\right| = \left|\vec{a} \times (\vec{b} \times K\vec{a})\right|$$
$$= \left|(\vec{a} \times \vec{b}) + K(\vec{a} \times \vec{a})\right| = \left|\vec{a} \times \vec{b}\right| + 0 \qquad [\because \vec{a} \times \vec{a} = 0]$$
$$= \left|\vec{a} \times \vec{b}\right|$$

Hence proved.

LONG ANSWER TYPE QUESTIONS

Q15. Prove that in any triangle ABC, $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, where a, b, c are the magnitudes of the sides opposite to the vertices

A, B, C respectively.



Sol. Here, in the given figure, the components of c are c cos A and c sin A.

$\therefore \quad \overrightarrow{CD} = b - c\cos A$

In ΔBDC,

$$a^2 = CD^2 + BD^2$$

$$\Rightarrow \quad a^2 = (b - c\cos A)^2 + (c\sin A)^2$$

$$\Rightarrow \quad a^2 = b^2 + c^2\cos^2 A - 2bc\cos A + c^2\sin^2 A$$

$$\Rightarrow \quad a^2 = b^2 + c^2(\cos^2 A + \sin^2 A) - 2bc\cos A$$

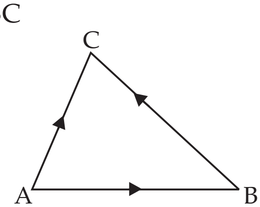
$$\Rightarrow \quad a^2 = b^2 + c^2 - 2bc\cos A \quad \Rightarrow 2bc\cos A = b^2 + c^2 - a^2$$

$$\therefore \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Hence Proved.

Q16. If $\vec{a}, \vec{b}$ and $\vec{c}$ determine the vertices of a triangle, show that $\frac{1}{2}[\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}]$ gives the vector area of the triangle. Hence deduce the condition that the three points $\vec{a}, \vec{b}$ and $\vec{c}$ are collinear. Also, find the unit vector normal to the plane of the triangle.

Sol. Since, $\vec{a}, \vec{b}$ and $\vec{c}$ are the vertices of ΔABC



$\therefore \quad \overrightarrow{AB} = \vec{b} - \vec{a}, \overrightarrow{BC} = \vec{c} - \vec{b}$

and $\quad \overrightarrow{AC} = \vec{c} - \vec{a}$

$$\therefore \quad \text{Area of } \Delta ABC = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$

$$= \frac{1}{2}\left|(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})\right|$$

$$= \frac{1}{2}\left|\vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c} + \vec{a} \times \vec{a}\right|$$

$$= \frac{1}{2}\left|\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a}\right| \qquad \left[\begin{array}{l} \because \quad \vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \\ \vec{c} \times \vec{a} = -\vec{a} \times \vec{c} \\ \vec{a} \times \vec{a} = \vec{0} \end{array}\right]$$

For three vectors are collinear, area of ΔABC = 0

$$\therefore \quad \frac{1}{2}\left|\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a}\right| = 0$$

$$\left|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}\right| = 0$$

which is the condition of collinearity of $\vec{a}, \vec{b}$ and $\vec{c}$.

Let $\hat{n}$ be the unit vector normal to the plane of the ΔABC

$$\therefore \quad \hat{n} = \frac{\overrightarrow{AB} \times \overrightarrow{AC}}{|\overrightarrow{AB} \times \overrightarrow{AC}|}$$

$$\Rightarrow \quad = \frac{\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}}{|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|}$$

Q17. Show that area of the parallelogram whose diagonals are given by $\vec{a}$ and $\vec{b}$ *is* $\frac{|\vec{a} \times \vec{b}|}{2}$. Also, find the area of the parallelogram, whose diagonals are $2\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} + 3\hat{j} - \hat{k}$.

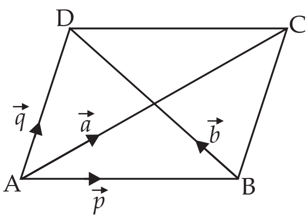
Sol. Let ABCD be a parallelogram such that,

$$\overrightarrow{AB} = \vec{p}, \overrightarrow{AD} = \vec{q} = \overrightarrow{BC}$$

$\therefore$ by law of triangle, we get

$$\overrightarrow{AC} = \vec{a} = \vec{p} + \vec{q} \quad ...(i)$$

and $\overrightarrow{BD} = \vec{b} = -\vec{p} + \vec{q} \quad ...(ii)$



Adding eq. (i) and (ii) we get,

$$\vec{a} + \vec{b} = 2\vec{q} \quad \Rightarrow \quad \vec{q} = \left(\frac{\vec{a} + \vec{b}}{2}\right)$$

Subtracting eq. (*ii*) from eq. (*i*) we get

$$\vec{a} - \vec{b} = 2\vec{p} \quad \Rightarrow \quad \vec{p} = \left(\frac{\vec{a} - \vec{b}}{2}\right)$$

$$\therefore \vec{p} \times \vec{q} = \frac{1}{4}(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \frac{1}{4}(\vec{a} \times \vec{a} - \vec{a} \times \vec{b} + \vec{b} \times \vec{a} - \vec{b} \times \vec{b})$$

$$= \frac{1}{4}(-\vec{a} \times \vec{b} + \vec{b} \times \vec{a}) \qquad \left[\because \begin{array}{l} \vec{a} \times \vec{a} = 0 \\ \vec{b} \times \vec{b} = 0 \end{array}\right]$$

$$= \frac{1}{4}(\vec{a} \times \vec{b} + \vec{a} \times \vec{b}) = \frac{1}{4} \cdot 2(\vec{a} \times \vec{b}) = \frac{|\vec{a} \times \vec{b}|}{2}$$

So, the area of the parallelogram ABCD $= |\vec{p} \times \vec{q}| = \frac{1}{2}|\vec{a} \times \vec{b}|$

Now area of parallelogram whose diagonals are $2\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} + 3\hat{j} - \hat{k} = \frac{1}{2}|(2\hat{i} - \hat{j} + \hat{k}) \times (\hat{i} + 3\hat{j} - \hat{k})|$

$$= -\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left|\hat{i}(1-3) - \hat{j}(-2-1) + \hat{k}(6+1)\right| = \frac{1}{2}\left|-2\hat{i} + 3\hat{j} + 7\hat{k}\right|$$

$$= \frac{1}{2}\sqrt{(-2)^2 + (3)^2 + (7)^2} = \frac{1}{2}\sqrt{4+9+49}$$

$$= \frac{1}{2}\sqrt{62} \text{ sq. units}$$

Hence, the required area is $\frac{1}{2}\sqrt{62}$ sq. units.

Q18. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$, find a vector $\vec{c}$ such that $\vec{a} \times \vec{c} = \vec{b}$ and $\vec{a} \cdot \vec{c} = 3$.

Sol. Let $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

Also given that $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$

Since, $\vec{a} \times \vec{c} = \vec{b}$

$$\therefore \quad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ c_1 & c_2 & c_3 \end{vmatrix} = \hat{j} - \hat{k}$$

$$= \hat{i}(c_3 - c_2) - \hat{j}(c_3 - c_1) + \hat{k}(c_2 - c_1) = \hat{j} - \hat{k}$$

On comparing the like terms, we get

$$c_3 - c_2 = 0 \qquad \text{...}(i)$$

$$c_1 - c_3 = 1 \qquad \text{...}(ii)$$

and $$c_2 - c_1 = -1 \qquad \text{...}(iii)$$

Now for $\vec{a} \cdot \vec{c} = 3$

$$(\hat{i} + \hat{j} + \hat{k}) \cdot (c_1\hat{i} + c_2\hat{j} + c_3\hat{k}) = 3$$

$$\therefore \quad c_1 + c_2 + c_3 = 3 \qquad \text{...}(iv)$$

Adding eq. (*ii*) and eq. (*iii*) we get,

$$c_2 - c_3 = 0 \qquad \text{...}(v)$$

From (*iv*) and (*v*) we get

$$c_1 + 2c_2 = 3 \qquad \text{...}(vi)$$

From (*iii*) and (*vi*) we get

$$\begin{array}{rr} & c_1 + 2c_2 = 3 \\ & -c_1 + c_2 = -1 \\ \hline \text{Adding} & 3c_2 = 2 \end{array}$$

$$\therefore \quad c_2 = \frac{2}{3}$$

$$c_3 - c_2 = 0 \quad \Rightarrow \quad c_3 - \frac{2}{3} = 0$$

$\therefore \qquad c_3 = \frac{2}{3}$

Now $\qquad c_2 - c_1 = -1 \quad \Rightarrow \quad \frac{2}{3} - c_1 = -1$

$\Rightarrow \qquad c_1 = 1 + \frac{2}{3} = \frac{5}{3}$

$\therefore \qquad \vec{c} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$

Hence, $\qquad \vec{c} = \frac{1}{3}(5\hat{i} + 2\hat{j} + 2\hat{k})$.

OBJECTIVE TYPE QUESTIONS

Choose the correct answer from the given four options in each of the Exercises from 19 to 33 (MCQ).

Q19. The vector in the direction of the vector $\hat{i} - 2\hat{j} + 2\hat{k}$ that has magnitude 9 is

(*a*) $\hat{i} - 2\hat{j} + 2\hat{k}$ (*b*) $\frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3}$

(*c*) $3(\hat{i} - 2\hat{j} + 2\hat{k})$ (*d*) $9(\hat{i} - 2\hat{j} + 2\hat{k})$

Sol. Let $\vec{a} = \hat{i} - 2\hat{j} + 2\hat{k}$

Unit vector in the direction of $\vec{a} = \frac{\vec{a}}{|\vec{a}|}$

$$= \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{\sqrt{(1)^2 + (-2)^2 + (2)^2}} = \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{\sqrt{1+4+4}} = \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3}$$

$\therefore$ Vector of magnitude $9 = \frac{9(\hat{i} - 2\hat{j} + 2\hat{k})}{3} = 3(\hat{i} - 2\hat{j} + 2\hat{k})$

Hence, the correct option is (*c*).

Q20. The position vector of the point which divides the join of points $2\vec{a} - 3\vec{b}$ and $\vec{a} + \vec{b}$ in the ratio $3:1$ is

(*a*) $\frac{3\vec{a} - 2\vec{b}}{2}$ (*b*) $\frac{7\vec{a} - 8\vec{b}}{4}$ (*c*) $\frac{3\vec{a}}{4}$ (*d*) $\frac{5\vec{a}}{4}$

Sol. The given vectors are $2\vec{a} - 3\vec{b}$ and $\vec{a} + \vec{b}$ and the ratio is 3 : 1.

$\therefore$ The position vector of the required point *c* which divides the join of the given vectors $\vec{a}$ and $\vec{b}$ is

$$\vec{c} = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}$$

$$= \frac{1 \cdot (2\vec{a} - 3\vec{b}) + 3(\vec{a} + \vec{b})}{3+1} = \frac{2\vec{a} - 3\vec{b} + 3\vec{a} + 3\vec{b}}{4}$$

$$= \frac{5\vec{a}}{4} = \frac{5}{4}\vec{a}$$

Hence, the correct option is (*d*).

Q21. The vector having initial and terminal points as (2, 5, 0) and (–3, 7, 4), respectively is

(*a*) $-\hat{i} + 12\hat{j} + 4\hat{k}$ (*b*) $5\hat{i} + 2\hat{j} - 4\hat{k}$

(*c*) $-5\hat{i} + 2\hat{j} + 4\hat{k}$ (*d*) $\hat{i} + \hat{j} + \hat{k}$

Sol. Let A and B be two points whose coordinates are given as (2, 5, 0) and (– 3, 7, 4)

$\therefore \quad \overrightarrow{AB} = (-3-2)\hat{i} + (7-5)\hat{j} + (4-0)\hat{k}$

$\Rightarrow \quad \overrightarrow{AB} = -5\hat{i} + 2\hat{j} + 4\hat{k}$

Hence, the correct option is (*c*).

Q22. The angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 4 respectively and $\vec{a} \cdot \vec{b} = 2\sqrt{3}$ is

(*a*) $\frac{\pi}{6}$ (*b*) $\frac{\pi}{3}$ (*c*) $\frac{\pi}{2}$ (*d*) $\frac{5\pi}{2}$

Sol. Here, given that $|\vec{a}| = \sqrt{3}, \left|\vec{b}\right| = 4$ and $\vec{a} \cdot \vec{b} = 2\sqrt{3}$

$\therefore$ From scalar product, we know that

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

$$\Rightarrow \quad 2\sqrt{3} = \sqrt{3} \cdot 4 \cdot \cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{2\sqrt{3}}{\sqrt{3} \cdot 4} = \frac{1}{2}$$

$$\therefore \quad \theta = \frac{\pi}{3}$$

Hence, the correct option is (*b*).

Q23. Find the value of λ such that the vectors $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$ are orthogonal

(*a*) 0 (*b*) 1 (*c*) $\frac{3}{2}$ (*d*) $\frac{-5}{2}$

Sol. Given that $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$

Since $\vec{a}$ and $\vec{b}$ are orthogonal

$$\therefore \quad \vec{a} \cdot \vec{b} = 0$$

$$\Rightarrow \quad (2\hat{i} + \lambda\hat{j} + \hat{k}) \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) = 0$$

$\Rightarrow \quad 2 + 2\lambda + 3 = 0$

$\Rightarrow \quad 5 + 2\lambda = 0 \quad \Rightarrow \quad \lambda = \frac{-5}{2}$

Hence, the correct option is (*d*).

Q24. The value of λ for which the vectors $3\hat{i} - 6\hat{j} + \hat{k}$ and $2\hat{i} - 4\hat{j} + \lambda\hat{k}$ are parallel is

(*a*) $\frac{2}{3}$ (*b*) $\frac{3}{2}$ (*c*) $\frac{5}{2}$ (*d*) $\frac{2}{5}$

Sol. Let $\vec{a} = 3\hat{i} - 6\hat{j} + \hat{k}$

$\vec{b} = 2\hat{i} - 4\hat{j} + \lambda\hat{k}$

Since the given vectors are parallel,

$\therefore$ Angle between them is $0°$

so $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos 0$

$\Rightarrow (3\hat{i} - 6\hat{j} + \hat{k}) \cdot (2\hat{i} - 4\hat{j} + \lambda\hat{k}) = |3\hat{i} - 6\hat{j} + \hat{k}||2\hat{i} - 4\hat{j} + \lambda\hat{k}|$

$6 + 24 + \lambda = \sqrt{9 + 36 + 1} \cdot \sqrt{4 + 16 + \lambda^2}$

$30 + \lambda = \sqrt{46} \cdot \sqrt{20 + \lambda^2}$

Squaring both sides, we get

$900 + \lambda^2 + 60\,\lambda = 46(20 + \lambda^2)$

$\Rightarrow \quad 900 + \lambda^2 + 60\lambda = 920 + 46\lambda^2$

$\Rightarrow \quad \lambda^2 - 46\lambda^2 + 60\lambda + 900 - 920 = 0$

$\Rightarrow \quad -45\lambda^2 + 60\lambda - 20 = 0$

$\Rightarrow \quad 9\lambda^2 - 12\lambda + 4 = 0$

$\Rightarrow \quad (3\lambda - 2)^2 = 0$

$\Rightarrow \quad 3\lambda - 2 = 0$

$\Rightarrow \quad 3\lambda = 2$

$\therefore \quad \lambda = 2/3$

Alternate method:

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

If $\vec{a} \parallel \vec{b}$

$\therefore \quad \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

$\Rightarrow \quad \frac{3}{2} = \frac{-6}{-4} = \frac{1}{\lambda} \quad \Rightarrow \quad \frac{1}{\lambda} = \frac{3}{2} \Rightarrow \lambda = \frac{2}{3}$

Hence, the correct option is (*a*).

Q25. The vectors from origin to the points A and B are

$\vec{a} = 2\hat{i} - 3\hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} + \hat{k}$ respectively, then the area of ΔOAB is equal to

(*a*) 340 (*b*) $\sqrt{25}$ (*c*) $\sqrt{229}$ (*d*) $\frac{1}{2}\sqrt{229}$

Sol. Let O be the origin

$$\therefore \quad \overrightarrow{OA} = 2\hat{i} - 3\hat{j} + 2\hat{k}$$

$$\text{and} \quad \overrightarrow{OB} = 2\hat{i} + 3\hat{j} + \hat{k}$$

$$\therefore \quad \text{Area of } \Delta OAB = \frac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{OB}\right| = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left|\hat{i}(-3-6) - \hat{j}(2-4) + \hat{k}(6+6)\right|$$

$$= \frac{1}{2}\left|-9\hat{i} + 2\hat{j} + 12\hat{k}\right|$$

$$= \frac{1}{2}\sqrt{(-9)^2 + (2)^2 + (12)^2}$$

$$= \frac{1}{2}\sqrt{81 + 4 + 144} = \frac{1}{2}\sqrt{229}$$

Hence the correct option is (*d*).

Q26. For any vector $\vec{a}$, the value of $(\vec{a} \times \hat{i})^2 + (\vec{a} \times \hat{j})^2 + (\vec{a} \times \hat{k})^2$ is

(*a*) $\vec{a}^2$ (*b*) $3\vec{a}^2$ (*c*) $4\vec{a}^2$ (*d*) $2\vec{a}^2$

Sol. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\therefore \quad \vec{a}^2 = a_1^2 + a_2^2 + a_3^2$$

$$\text{Now,} \quad \vec{a} \times \hat{i} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times \hat{i}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-a_3) + \hat{k}(0-a_2) = a_3\hat{j} - a_2\hat{k}$$

$$\therefore \quad (\vec{a} \times \hat{i})^2 = (a_3\hat{j} - a_2\hat{k}) \cdot (a_3\hat{j} - a_2\hat{k}) = a_3^2 + a_2^2$$

$$\text{Similarly} \quad (\vec{a} \times \hat{j})^2 = a_1^2 + a_3^2$$

$$\text{and} \quad (\vec{a} \times \hat{k})^2 = a_1^2 + a_2^2$$

$$\therefore \quad (\vec{a} \times \hat{i})^2 + (\vec{a} \times \hat{j})^2 + (\vec{a} \times \hat{k})^2 = a_3^2 + a_2^2 + a_1^2 + a_3^2 + a_1^2 + a_2^2$$

$$= 2(a_1^2 + a_2^2 + a_3^2) = 2\vec{a}^2$$

Hence, the correct option is (*d*).

Q27. If $|\vec{a}| = 10, |\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = 12$, then value of $|\vec{a} \times \vec{b}|$ is

(*a*) 5 (*b*) 10 (*c*) 14 (*d*) 16

Sol. Given that $|\vec{a}| = 10, |\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = 12$

$$\therefore \quad \vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$$

$$\Rightarrow \quad 12 = 10 \cdot 2 \cdot \cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{12}{20} = \frac{3}{5}$$

$$\therefore \quad \sin\theta = \sqrt{1 - \cos^2\theta}$$

$$\Rightarrow \quad \sin\theta = \sqrt{1 - \left(\frac{3}{5}\right)^2} \quad \Rightarrow \quad \sin\theta = \sqrt{1 - \frac{9}{25}}$$

$$\Rightarrow \quad \sin\theta = \sqrt{\frac{16}{25}} \quad \Rightarrow \quad \sin\theta = \frac{4}{5}$$

Now $$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$$

$$= 10 \cdot 2 \cdot \frac{4}{5} = 16$$

Hence, the correct option is (*d*).

Q28. The vectors $\lambda\hat{i} + \hat{j} + 2\hat{k}, \hat{i} + \lambda\hat{j} - \hat{k}$ and $2\hat{i} - \hat{j} + \lambda\hat{k}$ are coplanar if

(*a*) $\lambda = -2$ (*b*) $\lambda = 0$ (*c*) $\lambda = 1$ (*d*) $\lambda = -1$

Sol. Let $$\vec{a} = \lambda\hat{i} + \hat{j} + 2\hat{k}$$

$$\vec{b} = \hat{i} + \lambda\hat{j} - \hat{k}$$

$$\vec{c} = 2\hat{i} - \hat{j} + \lambda\hat{k}$$

If $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar, then

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$$

$$\therefore \quad \begin{vmatrix} \lambda & 1 & 2 \\ 1 & \lambda & -1 \\ 2 & -1 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda(\lambda^2 - 1) - 1\,(\lambda + 2) + 2(-1 - 2\lambda) = 0$$

$$\Rightarrow \quad \lambda^3 - \lambda - \lambda - 2 - 2 - 4\lambda = 0$$

$$\Rightarrow \quad \lambda^3 - 6\lambda - 4 = 0$$

$$\Rightarrow \quad (\lambda + 2)(\lambda^2 - 2\lambda - 2) = 0$$

$$\Rightarrow \quad \lambda = -2 \quad \text{or} \quad \lambda^2 - 2\lambda - 2 = 0$$

$$\Rightarrow \quad \lambda = \frac{2 \pm \sqrt{4+8}}{2}$$

$$\Rightarrow \quad \lambda = \frac{2 \pm 2\sqrt{3}}{2}$$

$$\therefore \quad \lambda = -2 \quad \text{or} \quad \lambda = 1 \pm \sqrt{3}$$

Hence, the correct option is (*a*).

Q29. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$ is

(*a*) 1 (*b*) 3 (*c*) $-\dfrac{3}{2}$ (*d*) None of these

Sol. Given that $|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$

and $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$$\therefore \quad (\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c}) = \vec{0} \cdot \vec{0} = 0$$

$$|\vec{a}|^2 + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{a} + |\vec{b}|^2 + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b} + |\vec{c}|^2 = 0$$

$$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{a} \cdot \vec{b} + 2\vec{b} \cdot \vec{c} + 2\vec{c} \cdot \vec{a} = 0$$

$$\Rightarrow \quad 1 + 1 + 1 + 2\,(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot a) = 0$$

$$\Rightarrow \quad 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = -3$$

$$\therefore \quad \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = \frac{-3}{2}$$

Hence, the correct option is (*c*).

Q30. The projection vector of $\vec{a}$ on $\vec{b}$ is :

(*a*) $\left(\dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}\right)\vec{b}$ (*b*) $\dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$ (*c*) $\dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$ (*d*) $\left(\dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2}\right)\vec{b}$

Sol. The projection vector of $\vec{a}$ on $\vec{b} = \left(\dfrac{\vec{a} \,.\, \vec{b}}{|\vec{b}|}\right) \cdot \vec{b}$

Hence, the correct option is (*a*).

Q31. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $|\vec{a}| = 2$, $|\vec{b}| = 3, |\vec{c}| = 5,$ then the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$ is

(*a*) 0 (*b*) 1 (*c*) -19 (*d*) 38

Sol. Given that $|\vec{a}| = 2,\ |\vec{b}| = 3, |\vec{c}| = 5,$

and $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$$(\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c}) = \vec{0} \cdot \vec{0} = 0$$

$$\Rightarrow |\vec{a}|^2 + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{a} + |\vec{b}|^2 + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b} + |\vec{c}|^2 = 0$$

$$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{a} \cdot \vec{b} + 2\vec{b} \cdot \vec{c} + 2\vec{c} \cdot \vec{a} = 0$$

$$\Rightarrow \quad (2)^2 + (3)^2 + (5)^2 + 2(\vec{a} . \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$$
$$\Rightarrow \quad 4 + 9 + 25 + 2\ (\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$$
$$\Rightarrow \quad 38 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$$
$$\Rightarrow \quad 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = -38$$
$$\therefore \quad \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = -19$$

Hence, the correct option is (*c*).

Q32. If $|\vec{a}| = 4$ and $-3 \le \lambda \le 2,$ then the range of $|\lambda\vec{a}|$ is

(*a*) [0, 8] (*b*) [– 12, 8] (*c*) [0, 12] (*d*) [8, 12]

Sol. Given that $|\vec{a}| = 4, -3 \le \lambda \le 2$

Now $|\lambda\vec{a}| = \lambda|\vec{a}| = \lambda \cdot 4 = 4\lambda$

Here $-3 \le \lambda \le 2$

$\Rightarrow \quad -3.4 \le 4\lambda \le 2.4 \quad \Rightarrow \quad -12 \le 4\lambda \le 8$

$\therefore \quad 4\lambda = [-12, 8]$

Hence, the correct option is (*b*).

Q33. The number of vectors of unit length perpendicular to the vectors $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{j} + \hat{k}$ is

(*a*) one (*b*) two (*c*) three (*d*) infinite

Sol. The number of vectors of unit length perpendicular to vectors $\vec{a}$ and $\vec{b}$ is $\vec{c}$ (let)

$$\therefore \quad \vec{c} = \pm(\vec{a} \times \vec{b})$$

So, there will be two vectors of unit length perpendicular to vectors $\vec{a}$ and $\vec{b}$.

Hence, the correct option is (*b*).

Fill in the blanks in each of the Exercises from 34 to 40.

Q34. The vector $\vec{a} + \vec{b}$ bisects the angle between the non-collinear vectors $\vec{a}$ and $\vec{b}$ if ______________ .

Sol. If vector $\vec{a} + \vec{b}$ bisects the angle between non-collinear vectors $\vec{a}$ and $\vec{b}$ then the angle between $\vec{a} + \vec{b}$ and $\vec{a}$ is equal to the angle between $\vec{a} + \vec{b}$ and $\vec{b}$.

So, $$\cos\theta = \frac{\vec{a} \cdot (\vec{a} + \vec{b})}{|\vec{a}||\vec{a} + \vec{b}|} = \frac{\vec{a} \cdot (\vec{a} + \vec{b})}{|\vec{a}|\sqrt{a^2 + b^2}} \quad ...(i)$$

Also, $$\cos\theta = \frac{\vec{b} \cdot (\vec{a} + \vec{b})}{|\vec{b}| \cdot |\vec{a} + \vec{b}|} \quad [\because \ \theta \text{ is same}]$$

$$= \frac{\vec{b} \cdot (\vec{a} + \vec{b})}{|\vec{b}|\sqrt{a^2 + b^2}} \quad ...(ii)$$

From eq. (*i*) and eq. (*ii*) we get,

$$\frac{\vec{a}\cdot(\vec{a}+\vec{b})}{|\vec{a}|\sqrt{a^2+b^2}} = \frac{\vec{b}\cdot(\vec{a}+\vec{b})}{|\vec{b}|\sqrt{a^2+b^2}}$$

$$\Rightarrow \quad \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{b}}{|\vec{b}|}$$

$$\Rightarrow \quad \hat{a} = \hat{b} \quad \Rightarrow \quad \vec{a} = \vec{b}$$

Hence, the required filler is $\vec{a} = \vec{b}$.

Q35. If $\vec{r}\cdot\vec{a} = 0, \vec{r}\cdot\vec{b} = 0$ and $\vec{r}\cdot\vec{c} = 0$ for some non-zero vector $\vec{r}$, then the value of $\vec{a}\cdot(\vec{b}\times\vec{c})$ is ____________ .

Sol. If $\vec{r}$ is a non-zero vector, then $\vec{a}$, $\vec{b}$ and $\vec{c}$ can be in the same plane.

Since angles between $\vec{a}$, and $\vec{c}$ are zero i.e. $\theta = 0$

$\therefore \quad \vec{a}\cdot(\vec{b}\times\vec{c}) = 0$

Hence the required value is 0.

Q36. The vectors $\vec{a} = 3\hat{i} - 2\hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} - 2\hat{k}$ are the adjacent sides of a parallelogram. The acute angle between its diagonals is ____________ .

Sol. Given that $\quad \vec{a} = 3\hat{i} - 2\hat{j} + 2\hat{k}$

and $\quad \vec{b} = -\hat{i} - 2\hat{k}$

$\therefore \quad \vec{a}+\vec{b} = 2\hat{i} - 2\hat{j}$ and $\vec{a}-\vec{b} = 4\hat{i} - 2\hat{j} + 4\hat{k}$

Let θ be the angle between the two diagonal vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ then

$$\cos\theta = \frac{(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})}{|\vec{a}+\vec{b}||\vec{a}-\vec{b}|} = \frac{(2\hat{i}-2\hat{j})\cdot(4\hat{i}-2\hat{j}+4\hat{k})}{\sqrt{(2)^2+(-2)^2}\cdot\sqrt{(4)^2+(-2)^2+(4)^2}}$$

$$= \frac{8+4}{2\sqrt{2}\cdot 6} = \frac{12}{2\sqrt{2}\cdot 6} = \frac{1}{\sqrt{2}}$$

$$\therefore \quad \theta = \frac{\pi}{4}$$

Hence the value of required filler is $\frac{\pi}{4}$.

Q37. The values of k, for which $|k\vec{a}| < |\vec{a}|$ and $k\vec{a} + \frac{1}{2}\vec{a}$ is parallel to $\vec{a}$ holds true are ____________ .

Sol. Given that $|k\vec{a}| < |\vec{a}|$ and $k\vec{a} + \frac{1}{2}\vec{a}$ is parallel to $\vec{a}$

$\therefore \quad |k\vec{a}| < |\vec{a}| \Rightarrow |k||\vec{a}| < |\vec{a}| \Rightarrow |k| < 1 \Rightarrow -1 < k < 1$

Now since $k\vec{a} + \frac{1}{2}\vec{a}$ is parallel to $\vec{a}$

Here we see that at $k = -\frac{1}{2}$, $k\vec{a} + \frac{1}{2}\vec{a}$ become null vector and then it will not be parallel to $\vec{a}$.

$\therefore$ $k\vec{a} + \frac{1}{2}\vec{a}$ is parallel to $\vec{a}$ when $k \in (-1, 1)$ and $k \neq \frac{1}{2}$.

Hence, the required value of $k \in (-1, 1)$ and $k \neq \frac{1}{2}$.

Q38. The value of the expression $\left|\vec{a} \times \vec{b}\right|^2 + (\vec{a} \cdot \vec{b})^2$ is ____________ .

Sol.
$$\begin{aligned}
\left|\vec{a} \times \vec{b}\right|^2 + (\vec{a} \cdot \vec{b})^2 &= \left(|\vec{a}|\left|\vec{b}\right| \sin\theta\right)^2 + \left(|\vec{a}|\left|\vec{b}\right| \cos\theta\right)^2 \\
&= |\vec{a}|^2 \left|\vec{b}\right|^2 \sin^2\theta + |\vec{a}|^2 \left|\vec{b}\right|^2 \cos^2\theta \\
&= |\vec{a}|^2 \left|\vec{b}\right|^2 \cdot (\sin^2\theta + \cos^2\theta) \\
&= |\vec{a}|^2 \left|\vec{b}\right|^2 \cdot 1 = |\vec{a}|^2 \left|\vec{b}\right|^2
\end{aligned}$$

Hence, the value of the filler is $|\vec{a}|^2 \left|\vec{b}\right|^2$.

Q39. If $\left|\vec{a} \times \vec{b}\right|^2 + \left|\vec{a} \cdot \vec{b}\right|^2 = 144$ and $|\vec{a}| = 4$, then $\left|\vec{b}\right|$ is equal to ____________ .

Sol.
$$\begin{aligned}
\left|\vec{a} \times \vec{b}\right|^2 + (\vec{a} \cdot \vec{b})^2 &= 144 \\
\Rightarrow \left(|\vec{a}|\left|\vec{b}\right| \sin\theta\right)^2 + \left(|\vec{a}|\left|\vec{b}\right| \cos\theta\right)^2 &= 144 \\
\Rightarrow |\vec{a}|^2 |\vec{b}|^2 \sin^2\theta + |\vec{a}|^2 |\vec{b}|^2 \cos^2\theta &= 144 \\
\Rightarrow |\vec{a}|^2 |\vec{b}|^2 (\sin^2\theta + \cos^2\theta) &= 144 \\
\Rightarrow |\vec{a}|^2 |\vec{b}|^2 &= 144 \\
\Rightarrow |\vec{a}|\,\left|\vec{b}\right| &= 12 \\
\Rightarrow 4 \cdot |\vec{b}| &= 12 \\
\therefore \left|\vec{b}\right| &= 3
\end{aligned}$$

Hence, the value of the filler is 3.

Q40. If $\vec{a}$ is any non-zero vector, then $(\vec{a} \cdot \hat{i})\hat{i} + (\vec{a} \cdot \hat{j})\hat{j} + (\vec{a} \cdot \hat{k})\hat{k}$ equals ____________ .

Sol. Let
$$\begin{aligned}
\vec{a} &= a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \\
\therefore \quad \vec{a} \cdot \hat{i} &= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot \hat{i} \\
&= a_1
\end{aligned}$$

Similarly, $\vec{a} \cdot \hat{j} = a_2$ and $\vec{a} \cdot \hat{k} = a_3$

$\therefore (\vec{a}\cdot\hat{i})\cdot\hat{i} + (\vec{a}\cdot\hat{j})\hat{j} + (\vec{a}\cdot\hat{k})\cdot\hat{k} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \vec{a}$

Hence, the value of the filler is $\vec{a}$.

State True or False in each of the following Exercises.

Q41. If $|\vec{a}| = |\vec{b}|$, then necessarily it implies $\vec{a} = \pm\vec{b}$.

Sol. If $|\vec{a}| = |\vec{b}|$ then $\vec{a} = \pm\vec{b}$ which is true.

Hence, the statement is True.

Q42. Position vector of a point P is a vector whose initial point is origin.

Sol. True

Q43. If $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$, then the vectors $\vec{a}$ and $\vec{b}$ are orthogonal.

Sol. Given that $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$

Squaring both sides, we get

$$|\vec{a}+\vec{b}|^2 = |\vec{a}-\vec{b}|^2$$

$$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b} = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}\cdot\vec{b}$$

$$\Rightarrow \quad 2\vec{a}\cdot\vec{b} = -2\vec{a}\cdot\vec{b} \Rightarrow \vec{a}\cdot\vec{b} = -\vec{a}\cdot\vec{b}$$

$$\Rightarrow \quad 2\vec{a}\cdot\vec{b} = 0 \Rightarrow \vec{a}\cdot\vec{b} = 0$$

which implies that $\vec{a}$ and $\vec{b}$ are orthogonal.

Hence the given statement is True.

Q44. The formula $(\vec{a}+\vec{b})^2 = \vec{a}^2 + \vec{b}^2 + 2\vec{a}\times\vec{b}$ is valid for non-zero vectors $\vec{a}$ and $\vec{b}$.

Sol.

$$(\vec{a}+\vec{b})^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$$

$$= |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b}$$

Hence, the given statement is False.

Q45. If $\vec{a}$ and $\vec{b}$ are adjacent sides of a rhombus, then $\vec{a}\cdot\vec{b} = 0$.

Sol. If $\vec{a}\cdot\vec{b} = 0$ then $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos 90°$

So the angle between the adjacent sides of the rhombus should be 90° which is not possible.

Hence, the given statement is False.